Math 280: 7.5 Strategy for Integration
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Some Basic Identities
$\cos^2 x + \sin^2 x = 1$
$1 + \tan^2 x = \sec^2 x$
$1 + \cot^2 x = \csc^2 x$
$\sin 2x = 2\sin x \cos x$
$\cos^2 x = \frac{1}{2}(1+\cos 2x)$
$\sin^2 x = \frac{1}{2}(1-\cos 2x)$

1. **Simplify the integrand, if possible. Do a little algebra first.**
   - Distribute, multiply out (FOIL), etc.
   - Rationalize the numerator (or denominator)
   - Change trig functions to sines and cosines
   - Use a basic trig identity
2. **Look for an obvious substitution.**
   - Look for a $u = g(x)$ for which $du = g'(x)dx$ is readily available (except for a constant multiple)
   - If possible, attempt substitution BEFORE partial fractions or integration by parts
3. **Classify the integrand**
   - Trigonometric Forms: powers of sinx and cosx, powers of tanx and secx
   - Rational Function: try Partial Fractions
   - A product of functions (or single term): Integration by Parts $\int u\,dv = uv - \int v\,du$
   - Radicals: if $\sqrt{\pm x^2 \pm a^2}$, use Trig Substitution →
     if $\sqrt[n]{ax+b}$, try $u = \sqrt[n]{ax+b}$
   - Powers: if $\left(\pm x^2 \pm a^2\right)^{m/n}$, use Trig Substitution →

Expression in the integrand	*Substitution*
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{x^2 + a^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

4. **Try, Try again. Persevere!**
   - Try Substitution
   - Try Integration by Parts
   - Manipulate the integrand with algebra
   - Relate the problem to a previous problem
   - If the problem reverts back to the original integral, add/subtract the integral to the both sides of the equation to solve for the unknown integral
   - Use more than one method. Separate sums/differences and look at the terms as separate integration problems. Whittle away at the integral!
   - Persevere!

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1. For each of the following integrals, name one integration technique you could use to evaluate the integral effectively. DO NOT INTEGRATE!!

$\int \frac{dx}{\left(1-x^2\right)^{3/2}}$	$\int x^2 e^{x^3}\,dx$	$\int \frac{dx}{x^2-4x-5}$	$\int x\cos x\,dx$
Trig. Substitution $x = \sin\theta$	u-substitution $u = x^3$	partial Fractions $\frac{A}{x-5} + \frac{B}{x+1}$	By Parts $u = x$, $dv = \cos x$

looks like $\sin 2t = 2\sin t \cos t$

2. $\int t \sin t \cos t \, dt = \int t \cdot \frac{1}{2}(2\sin t \cos t)\,dt = \frac{1}{2}\int t \sin 2t \, dt$

$$\boxed{\begin{array}{ll} u = t & v = -\frac{1}{2}\cos 2t \\ du = dt & dv = \sin 2t\,dt \end{array}}$$

$$= \frac{1}{2}\left[-\frac{1}{2}t\cos 2t + \frac{1}{2}\int \cos 2t\,dt\right]$$

$$= -\frac{1}{4}t\cos 2t + \frac{1}{4}\cdot\frac{1}{2}\sin 2t + C$$

$$= -\frac{1}{4}t\cos 2t + \frac{1}{8}\sin 2t + C$$

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3. $\int (1+\sqrt{x})^8 \, dx$

$u = 1+\sqrt{x}$ $\qquad x = (u-1)^2$

$dx = 2(u-1)\,du$

$$= \int u^8 \cdot 2(u-1)\,du$$

$$= 2\int u^9 - u^8 \, du$$

$$= 2\left(\frac{u^{10}}{10} - \frac{u^9}{9}\right) + C$$

$$= 2\left(\frac{1}{10}(1+\sqrt{x})^{10} - \frac{1}{9}(1+\sqrt{x})^9\right) + C$$

$$= \frac{1}{5}(1+\sqrt{x})^{10} - \frac{2}{9}(1+\sqrt{x})^9 + C$$

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4. $\int e^{x+e^x} \, dx = \int e^x \cdot e^{e^x}\,dx$

$u = e^x$

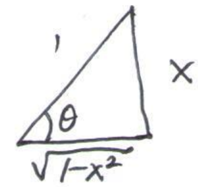
$du = e^x dx$

$$= \int e^u \, du$$

$$= e^u + C = e^{e^x} + C$$

5. $\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx$ } Break up integral

$= \underbrace{\int \frac{1}{\sqrt{1-x^2}}} + \int \frac{x}{\sqrt{1-x^2}}$ } trig. sub. $x = \sin\theta$, $dx = \cos\theta\, d\theta$



$= \sin^{-1} x + \int \frac{\sin\theta}{\sqrt{1-\sin^2\theta}} \cdot \cos\theta\, d\theta$

$= \sin^{-1} x + \int \frac{\sin\theta}{\cancel{\cos\theta}} \cdot \cancel{\cos\theta}\, d\theta = \sin^{-1} x + \int \sin\theta\, d\theta = \sin^{-1} x - \cos\theta + C$

$= \sin^{-1} x - \sqrt{1-x^2} + C$

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6. $\int \frac{\sec\theta\tan\theta}{\sec^2\theta - \sec\theta}\,d\theta$ $\left\{ \begin{array}{l} u = \sec\theta \\ du = \sec\theta\tan\theta\, d\theta \end{array} \right\} = \int \frac{du}{u^2 - u} = \int \frac{1}{u(u-1)}\,du$

Partial fractions

$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1}$

$1 = A(u-1) + B(u)$

$1 = Au - A + Bu$

$\left.\begin{array}{l} 0 = A + B \\ 1 = -A \end{array}\right\}$ $\begin{array}{l} A = -1 \\ B = 1 \end{array}$

$\frac{1}{u(u-1)} = \frac{-1}{u} + \frac{1}{u-1}$

$= \int \left(\frac{-1}{u} + \frac{1}{u-1} \right) du$

$= -\ln|u| + \ln|u-1| + C$

$= \ln|\sec\theta - 1| - \ln|\sec\theta| + C$

$= \ln\left| \frac{\sec\theta - 1}{\sec\theta} \right| + C = \ln|1 - \cos\theta| + C$

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7. $\int \theta \tan^2\theta\, d\theta$ } By Parts $\begin{array}{l} u = \theta \\ du = d\theta \end{array}$ $\begin{array}{l} v = \tan\theta - \theta \\ dv = \tan^2\theta\, d\theta \\ dv = (\sec^2\theta - 1)\,d\theta \end{array}$

$= \theta(\tan\theta - \theta) - \int (\tan\theta - \theta)\,d\theta$

$= \theta\tan\theta - \theta^2 - \left[\ln|\sec\theta| - \frac{1}{2}\theta^2 \right] + C$

$= \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \frac{1}{2}\theta^2 + C$

$= \theta\tan\theta - \frac{1}{2}\theta^2 - \ln|\sec\theta| + C$

$= \theta\tan\theta - \frac{1}{2}\theta^2 + \ln|\cos\theta| + C$