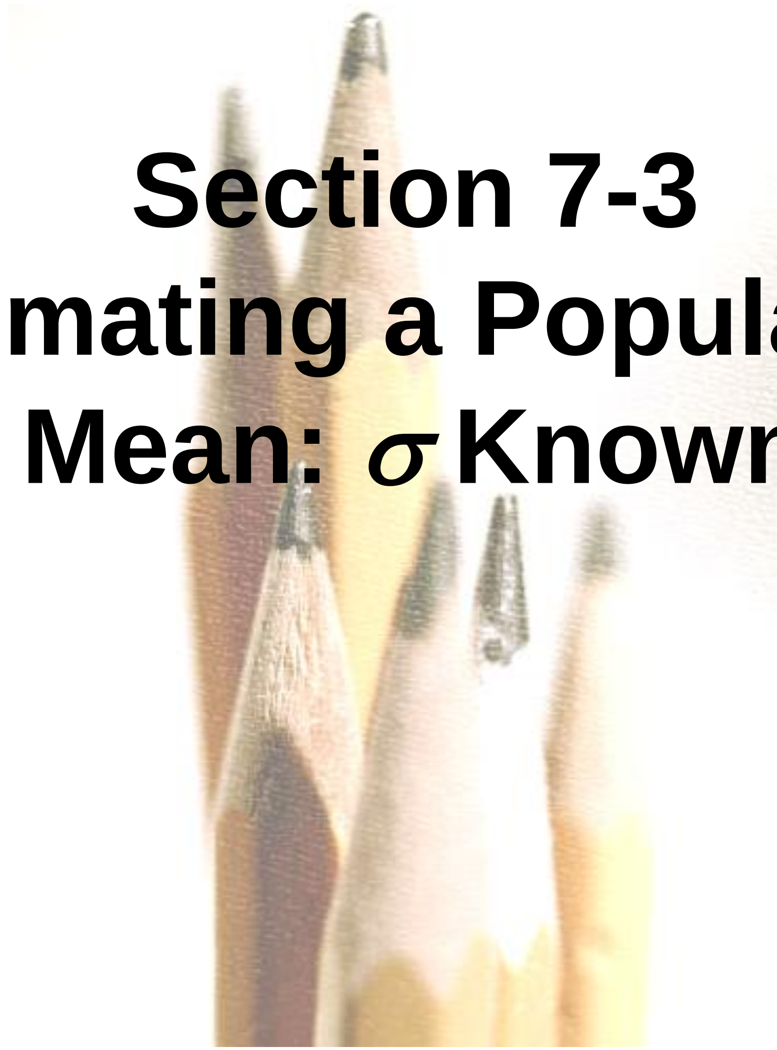
Chapter 7
Estimates and Sample Sizes

Section 7-3
Estimating a Population Mean: σ Known



Key Concept

This section presents methods for estimating a population mean. In addition to knowing the values of the sample data or statistics, we must also know the value of the population standard deviation, σ.

Here are three key concepts that should be learned in this section:

Key Concept

1. We should know that the sample mean $\bar{x}$ is the best point estimate of the population mean μ.

2. We should learn how to use sample data to construct a confidence interval for estimating the value of a population mean, and we should know how to interpret such confidence intervals.

3. We should develop the ability to determine the sample size necessary to estimate a population mean.

Point Estimate of the Population Mean

The sample mean $\bar{x}$ is the best point estimate of the population mean μ.

Confidence Interval for Estimating a Population Mean (with σ Known)

μ = population mean

σ = population standard deviation

$\bar{x}$ = sample mean

n = number of sample values

E = margin of error

$z_{\alpha/2}$ = z score separating an area of $a/2$ in the right tail of the standard normal distribution

Confidence Interval for Estimating a Population Mean (with σ Known)

1. The sample is a simple random sample. (All samples of the same size have an equal chance of being selected.)

2. The value of the population standard deviation σ is known.

3. Either or both of these conditions is satisfied: The population is normally distributed or $n > 30$.

Confidence Interval for Estimating a Population Mean (with σ Known)

$$\bar{x} - E < \mu < \bar{x} + E \quad \text{where} \quad E = z_{\alpha/2} \cdot \frac{\sigma}{\sqrt{n}}$$

or $$\bar{x} \pm E$$

or $$(\bar{x} - E, \bar{x} + E)$$

Definition

The two values $\bar{x}$ – *E* and $\bar{x}$ + *E* are called confidence interval limits.

Sample Mean

1. For all populations, the sample mean $\bar{x}$ is an unbiased estimator of the population mean μ, meaning that the distribution of sample means tends to center about the value of the population mean μ.

2. For many populations, the distribution of sample means $\bar{x}$ tends to be more consistent (with less variation) than the distributions of other sample statistics.

Procedure for Constructing a Confidence Interval for μ (with Known σ)

1. Verify that the requirements are satisfied.
2. Refer to Table A-2 or use technology to find the critical value $z_{\alpha/2}$ that corresponds to the desired confidence level.
3. Evaluate the margin of error $E = z_{\alpha/2} \cdot \sigma / \sqrt{n}$
4. Find the values of $\bar{x} - E$ and $\bar{x} + E$. Substitute those values in the general format of the confidence interval:

$$\bar{x} - E < \mu < \bar{x} + E$$

5. Round using the confidence intervals round-off rules.

Round-Off Rule for Confidence Intervals Used to Estimate μ

1. When using the original set of data, round the confidence interval limits to one more decimal place than used in original set of data.
2. When the original set of data is unknown and only the summary statistics (n, $\bar{x}$, s) are used, round the confidence interval limits to the same number of decimal places used for the sample mean.

Example:

People have died in boat and aircraft accidents because an obsolete estimate of the mean weight of men was used. In recent decades, the mean weight of men has increased considerably, so we need to update our estimate of that mean so that boats, aircraft, elevators, and other such devices do not become dangerously overloaded. Using the weights of men from Data Set 1 in Appendix B, we obtain these sample statistics for the simple random sample: n = 40 and $\bar{x}$ = 172.55 lb. Research from several other sources suggests that the population of weights of men has a standard deviation given by σ = 26 lb.

Example:

a. Find the best point estimate of the mean weight of the population of all men.

b. Construct a 95% confidence interval estimate of the mean weight of all men.

c. What do the results suggest about the mean weight of 166.3 lb that was used to determine the safe passenger capacity of water vessels in 1960 (as given in the National Transportation and Safety Board safety recommendation M-04-04)?

Example:

a. **The sample mean of 172.55 lb is the best point estimate of the mean weight of the population of all men.**

b. **A 95% confidence interval or 0.95 implies σ = 0.05, so $z_{\alpha/2}$ = 1.96.**
Calculate the margin of error.

$$E = z_{\alpha/2} \bullet \frac{\sigma}{\sqrt{n}} = 1.96 \bullet \frac{26}{\sqrt{40}} = 8.0574835$$

Construct the confidence interval.

$$\bar{x} - E < \mu < \bar{x} - E$$

$$172.55 - 8.0574835 < \mu < 172.55 + 8.0574835$$

$$164.49 < \mu < 180.61$$

Example:

c. Based on the confidence interval, it is possible that the mean weight of 166.3 lb used in 1960 could be the mean weight of men today. However, the best point estimate of 172.55 lb suggests that the mean weight of men is now considerably greater than 166.3 lb. Considering that an underestimate of the mean weight of men could result in lives lost through overloaded boats and aircraft, these results strongly suggest that additional data should be collected. (Additional data have been collected, and the assumed mean weight of men has been increased.)

Finding a Sample Size for Estimating a Population Mean

μ = population mean

σ = population standard deviation

$\bar{x}$ = population standard deviation

E = desired margin of error

$z_{\alpha/2}$ = z score separating an area of $\alpha/2$ in the right tail of the standard normal distribution

$$n = \left[\frac{(z_{\alpha/2}) \cdot \sigma}{E}\right]^2$$

Round-Off Rule for Sample Size *n*

If the computed sample size *n* is not a whole number, round the value of *n* up to the next larger whole number.

Finding the Sample Size *n* When σ is Unknown

1. Use the range rule of thumb (see Section 3-3) to estimate the standard deviation as follows: $\sigma \approx \text{range}/4$.
2. Start the sample collection process without knowing σ and, using the first several values, calculate the sample standard deviation s and use it in place of σ. The estimated value of σ can then be improved as more sample data are obtained, and the sample size can be refined accordingly.
3. Estimate the value of σ by using the results of some other study that was done earlier.

Example:

Assume that we want to estimate the mean IQ score for the population of statistics students. How many statistics students must be randomly selected for IQ tests if we want 95% confidence that the sample mean is within 3 IQ points of the population mean?

$\alpha = 0.05$

$\alpha/2 = 0.025$

$z_{\alpha/2} = 1.96$

$E = 3$

$\sigma = 15$

$$n = \left[\frac{1.96 \cdot 15}{3}\right]^2 = 96.04 = 97$$

With a simple random sample of only 97 statistics students, we will be 95% confident that the sample mean is within 3 IQ points of the true population mean μ.

Recap

In this section we have discussed:

- **Margin of error.**
- **Confidence interval estimate of the population mean with σ known.**
- **Round off rules.**
- **Sample size for estimating the mean μ.**

Section 7-4
Estimating a Population Mean: σ Not Known

Key Concept

This section presents methods for estimating a population mean when the population standard deviation is not known. With *σ* unknown, we use the Student *t* distribution assuming that the relevant requirements are satisfied.

Student *t* Distribution

If the distribution of a population is essentially normal, then the distribution of

$$t = \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}}$$

is a Student *t* Distribution for all samples of size *n*. It is often referred to as a *t* distribution and is used to find critical values denoted by $t_{\alpha/2}$.

Definition

The number of degrees of freedom for a collection of sample data is the number of sample values that can vary after certain restrictions have been imposed on all data values. The degree of freedom is often abbreviated df.

degrees of freedom = n – 1

in this section.

Margin of Error E for Estimate of μ (With σ Not Known)

Formula 7-6

$$E = t_{\alpha/2} \frac{s}{\sqrt{n}}$$

where $t_{\alpha/2}$ has $n - 1$ degrees of freedom.

Table A-3 lists values for $t_{\alpha/2}$

Notation

μ = population mean

$\bar{x}$ = sample mean

s = sample standard deviation

n = number of sample values

E = margin of error

$t_{\alpha/2}$ = critical t value separating an area of $\alpha/2$ in the right tail of the t distribution

Confidence Interval for the Estimate of μ (With σ Not Known)

$$\bar{x} - E < \mu < \bar{x} + E$$

where $$E = t_{\alpha/2} \frac{s}{\sqrt{n}}$$ df = n – 1

$t_{\alpha/2}$ found in Table A-3

Procedure for Constructing a Confidence Interval for μ (With σ Unknown)

1. Verify that the requirements are satisfied.

2. Using $n - 1$ degrees of freedom, refer to Table A-3 or use technology to find the critical value $t_{\alpha/2}$ that corresponds to the desired confidence level.

3. Evaluate the margin of error $E = t_{\alpha/2} \cdot s / \sqrt{n}$.

4. Find the values of $\bar{x} - E$ and $\bar{x} + E$. Substitute those values in the general format for the confidence interval:

$$\bar{x} - E < \mu < \bar{x} + E$$

5. Round the resulting confidence interval limits.

Example:

A common claim is that garlic lowers cholesterol levels. In a test of the effectiveness of garlic, 49 subjects were treated with doses of raw garlic, and their cholesterol levels were measured before and after the treatment. The changes in their levels of LDL cholesterol (in mg/dL) have a mean of 0.4 and a standard deviation of 21.0. Use the sample statistics of n = 49, $\overline{x}$ = 0.4 and s = 21.0 to construct a 95% confidence interval estimate of the mean net change in LDL cholesterol after the garlic treatment. What does the confidence interval suggest about the effectiveness of garlic in reducing LDL cholesterol?

Example:

Requirements are satisfied: simple random sample and n = 49 (i.e., n > 30).

95% implies a = 0.05.
With n = 49, the df = 49 – 1 = 48
Closest df is 50, two tails, so $t_{\alpha/2}$ = 2.009

Using $t_{\alpha/2}$ = 2.009, s = 21.0 and n = 49 the margin of error is:

$$E = t_{\alpha/2} \bullet \frac{\sigma}{\sqrt{n}} = 2.009 \bullet \frac{21.0}{\sqrt{49}} = 6.027$$

Example:

Construct the confidence interval:

$$\bar{x} = 0.4, E = 6.027$$

$$\bar{x} - E < \mu < \bar{x} + E$$

$$0.4 - 6.027 < \mu < 0.4 + 6.027$$

$$-5.6 < \mu < 6.4$$

We are 95% confident that the limits of –5.6 and 6.4 actually do contain the value of μ, the mean of the changes in LDL cholesterol for the population. Because the confidence interval limits contain the value of 0, it is very possible that the mean of the changes in LDL cholesterol is equal to 0, suggesting that the garlic treatment did not affect the LDL cholesterol levels. It does not appear that the garlic treatment is effective in lowering LDL cholesterol.

Important Properties of the Student *t* Distribution

1. The Student *t* distribution is different for different sample sizes (see the following slide, for the cases $n = 3$ and $n = 12$).

2. The Student *t* distribution has the same general symmetric bell shape as the standard normal distribution but it reflects the greater variability (with wider distributions) that is expected with small samples.

3. The Student *t* distribution has a mean of $t = 0$ (just as the standard normal distribution has a mean of $z = 0$).

4. The standard deviation of the Student *t* distribution varies with the sample size and is greater than 1 (unlike the standard normal distribution, which has a $\sigma = 1$).

5. As the sample size *n* gets larger, the Student *t* distribution gets closer to the normal distribution.

Student *t* Distributions for *n* = 3 and *n* = 12

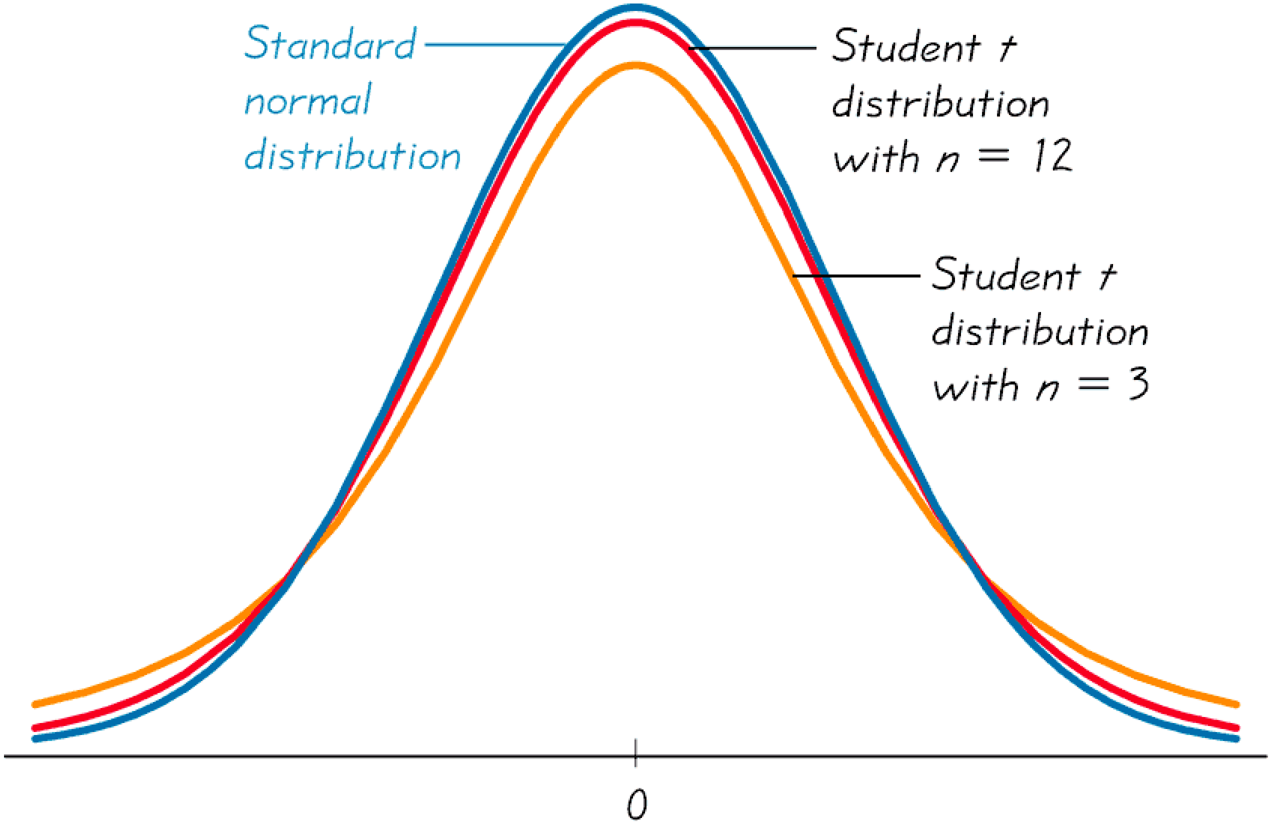
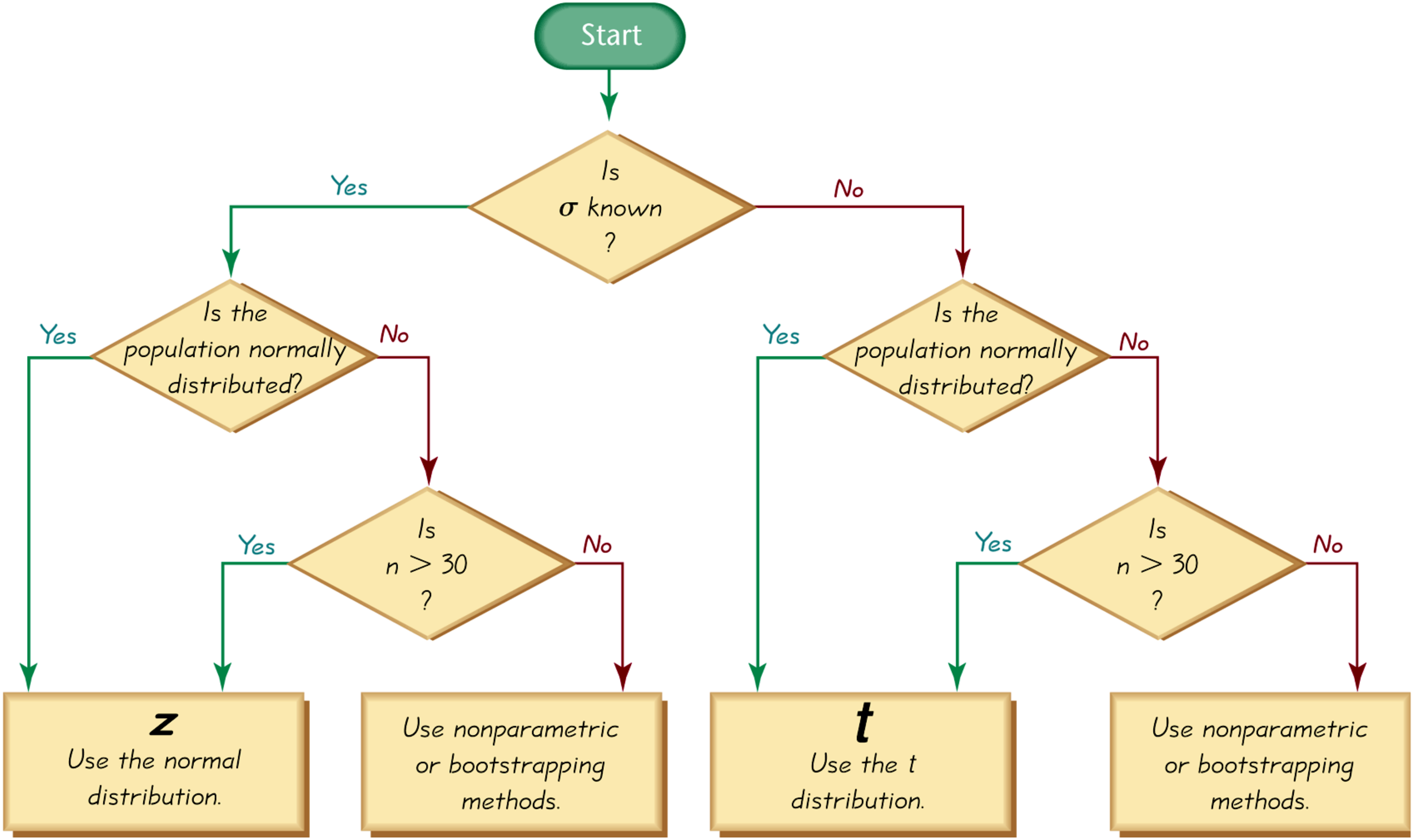


Figure 7-5

Choosing the Appropriate Distribution



Choosing the Appropriate Distribution

Use the normal (z) distribution	σ known and normally distributed population or σ known and $n > 30$
Use t distribution	σ not known and normally distributed population or σ not known and $n > 30$
Use a nonparametric method or bootstrapping	Population is not normally distributed and $n \leq 30$

Finding the Point Estimate and E from a Confidence Interval

Point estimate of μ:

$$\bar{x} = \frac{\text{(upper confidence limit)} + \text{(lower confidence limit)}}{2}$$

Margin of Error:

$$E = \frac{\text{(upper confidence limit)} - \text{(lower confidence limit)}}{2}$$

Confidence Intervals for Comparing Data

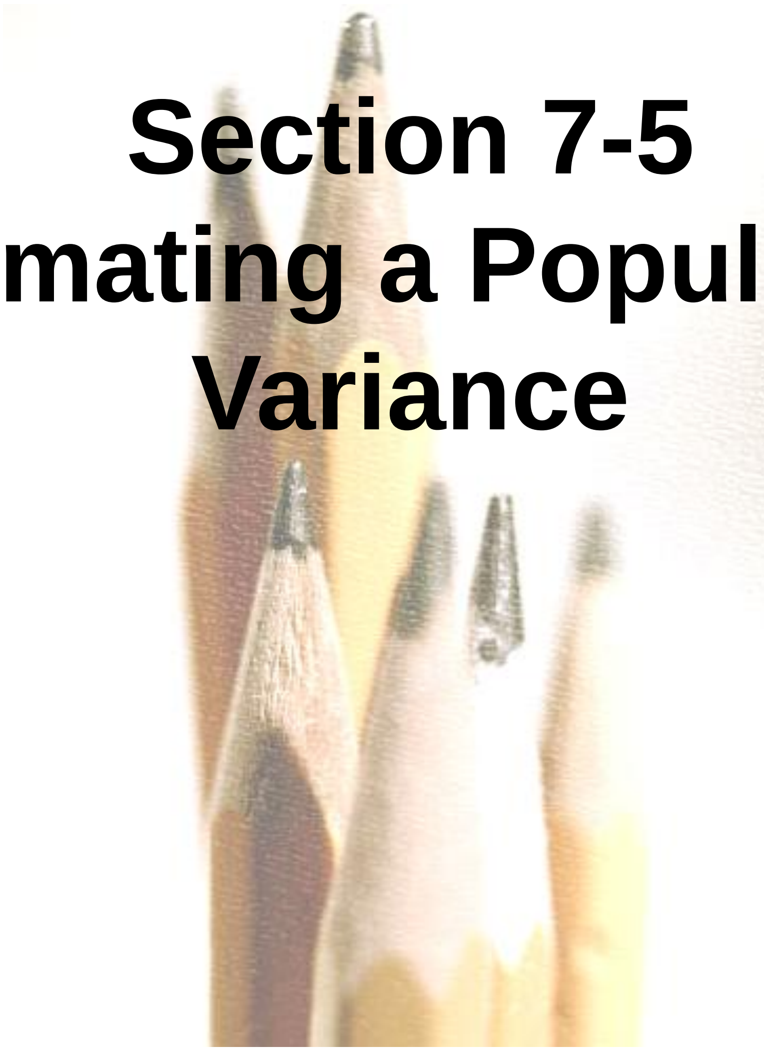
As in Sections 7-2 and 7-3, confidence intervals can be used informally to compare different data sets, but the overlapping of confidence intervals should not be used for making formal and final conclusions about equality of means.

Recap

In this section we have discussed:

- Student *t* distribution.
- Degrees of freedom.
- Margin of error.
- Confidence intervals for μ with σ unknown.
- Choosing the appropriate distribution.
- Point estimates.
- Using confidence intervals to compare data.

Section 7-5
Estimating a Population Variance



Key Concept

This section we introduce the chi-square probability distribution so that we can construct confidence interval estimates of a population standard deviation or variance. We also present a method for determining the sample size required to estimate a population standard deviation or variance.

Chi-Square Distribution

In a normally distributed population with variance σ^2 assume that we randomly select independent samples of size *n* and, for each sample, compute the sample variance s^2 (which is the square of the sample standard deviation s). The sample statistic χ^2 (pronounced chi-square) has a sampling distribution called the chi-square distribution.

Chi-Square Distribution

$$\chi^2 = \frac{(n-1)\,s^2}{\sigma^2}$$

where

n = sample size

s^2 = sample variance

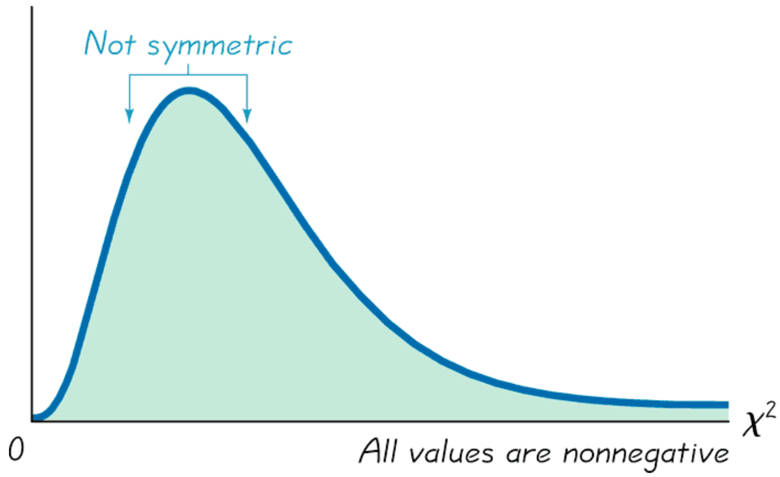
σ^2 = population variance

degrees of freedom = $n - 1$

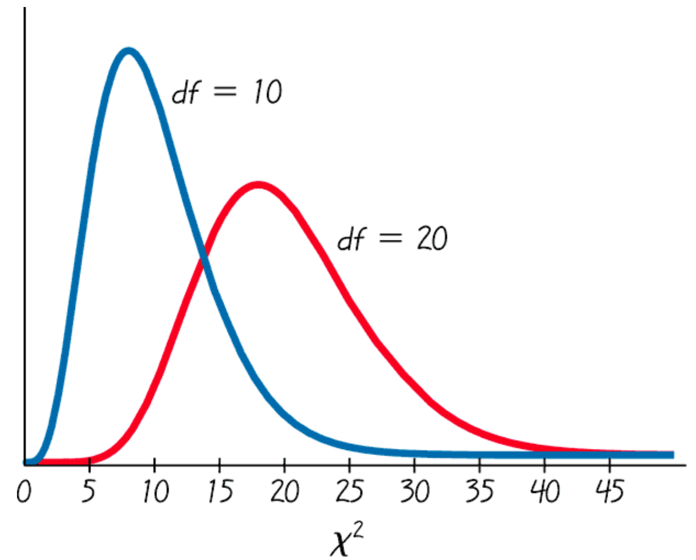
Properties of the Distribution of the Chi-Square Statistic

1. **The chi-square distribution is not symmetric, unlike the normal and Student *t* distributions.**

   As the number of degrees of freedom increases, the distribution becomes more symmetric.



Chi-Square Distribution



Chi-Square Distribution for df = 10 and df = 20

Properties of the Distribution of the Chi-Square Statistic – cont.

2. **The values of chi-square can be zero or positive, but they cannot be negative.**

3. **The chi-square distribution is different for each number of degrees of freedom, which is df = n – 1. As the number of degrees of freedom increases, the chi-square distribution approaches a normal distribution.**

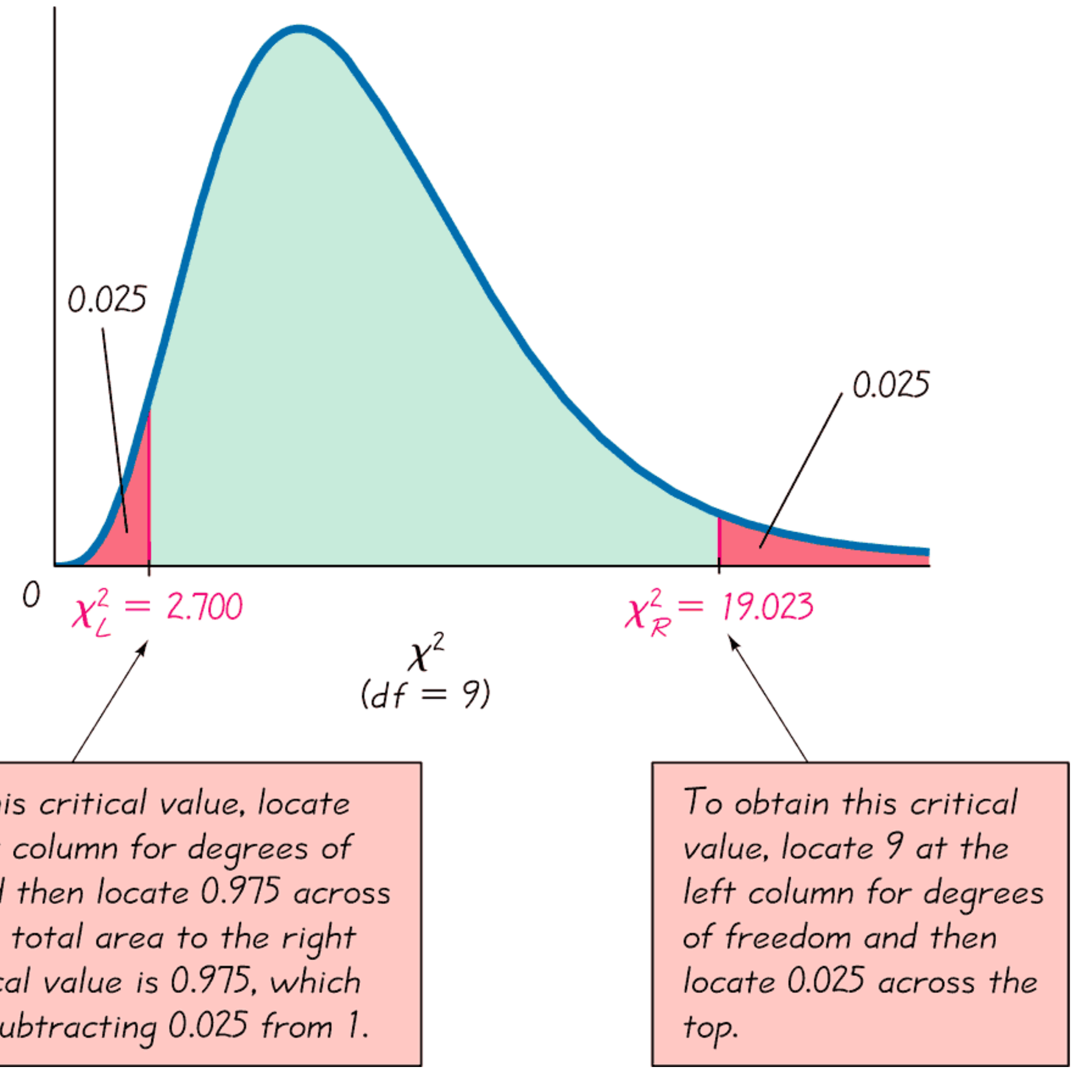
   In Table A-4, each critical value of χ^2 corresponds to an area given in the top row of the table, and that area represents the cumulative area located to the right of the critical value.

Example

A simple random sample of ten voltage levels is obtained. Construction of a confidence interval for the population standard deviation σ requires the left and right critical values of χ^2 corresponding to a confidence level of 95% and a sample size of n = 10. Find the critical value of χ^2 separating an area of 0.025 in the left tail, and find the critical value of χ^2 separating an area of 0.025 in the right tail.

Example

Critical Values of the Chi-Square Distribution



Example

For a sample of 10 values taken from a normally distributed population, the chi-square statistic $\chi^2 = (n - 1)s^2/\sigma^2$ has a 0.95 probability of falling between the chi-square critical values of 2.700 and 19.023.

Instead of using Table A-4, technology (such as STATDISK, Excel, and Minitab) can be used to find critical values of χ^2. A major advantage of technology is that it can be used for any number of degrees of freedom and any confidence level, not just the limited choices included in Table A-4.

Estimators of σ^2

The sample variance s^2 is the best point estimate of the population variance σ^2.

Estimators of σ

The sample standard deviation *s* is a commonly used point estimate of σ (even though it is a biased estimate).

Confidence Interval for Estimating a Population Standard Deviation or Variance

σ = population standard deviation

s = sample standard deviation

n = number of sample values

χ^2_L = left-tailed critical value of χ^2

σ^2 = population variance

s^2 = sample variance

E = margin of error

χ^2_R = right-tailed critical value of χ^2

Confidence Interval for Estimating a Population Standard Deviation or Variance

Requirements:

1. The sample is a simple random sample.

2. The population must have normally distributed values (even if the sample is large).

Confidence Interval for Estimating a Population Standard Deviation or Variance

Confidence Interval for the Population Variance σ^2

$$\frac{(n-1)s^2}{\chi_R^2} < \sigma^2 < \frac{(n-1)s^2}{\chi_L^2}$$

Confidence Interval for Estimating a Population Standard Deviation or Variance

Confidence Interval for the Population Standard Deviation σ

$$\sqrt{\frac{(n-1)s^2}{\chi_R^2}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi_L^2}}$$

Procedure for Constructing a Confidence Interval for σ or σ^2

1. **Verify that the required assumptions are satisfied.**

2. **Using $n - 1$ degrees of freedom, refer to Table A-4 or use technology to find the critical values χ^2_R and χ^2_L that correspond to the desired confidence level.**

3. **Evaluate the upper and lower confidence interval limits using this format of the confidence interval:**

$$\frac{(n-1)s^2}{\chi^2_R} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_L}$$

Procedure for Constructing a Confidence Interval for σ or σ^2 - cont

4. If a confidence interval estimate of σ is desired, take the square root of the upper and lower confidence interval limits and change σ^2 to σ.

5. Round the resulting confidence level limits. If using the original set of data to construct a confidence interval, round the confidence interval limits to one more decimal place than is used for the original set of data. If using the sample standard deviation or variance, round the confidence interval limits to the same number of decimals places.

Confidence Intervals for Comparing Data
Caution

Confidence intervals can be used informally to compare the variation in different data sets, but the overlapping of confidence intervals should not be used for making formal and final conclusions about equality of variances or standard deviations.

Example:

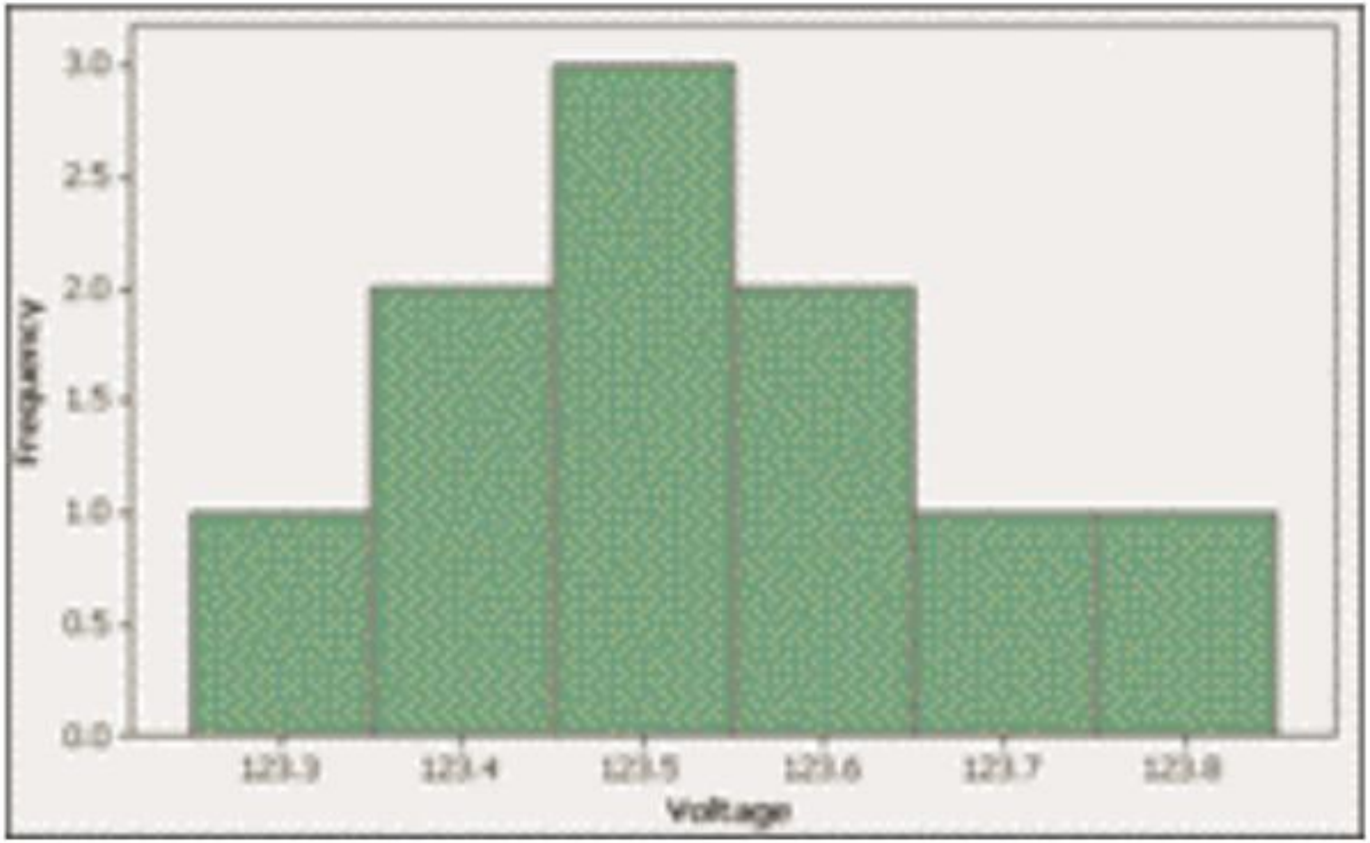
The proper operation of typical home appliances requires voltage levels that do not vary much. Listed below are ten voltage levels (in volts) recorded in the author's home on ten different days. These ten values have a standard deviation of $s = 0.15$ volt. Use the sample data to construct a 95% confidence interval estimate of the standard deviation of all voltage levels.

123.3 123.5 123.7 123.4 123.6 123.5 123.5 123.4 123.6 123.8

Example:

Requirements are satisfied: simple random sample and normality

MINITAB



Example:

n **= 10 so df = 10 – 1 = 9**

Use table A-4 to find:

$$\chi_L^2 = 2.700 \text{ and } \chi_R^2 = 19.023$$

Construct the confidence interval: *n* = 10, *s* = 0.15

$$\frac{(n-1)s^2}{\chi_R^2} < \sigma^2 < \frac{(n-1)s^2}{\chi_L^2}$$

$$\frac{(10-1)(0.15)^2}{19.023} < \sigma^2 < \frac{(10-1)(0.15)^2}{2.700}$$

Example:

Evaluation the preceding expression yields:

$$0.010645 < \sigma^2 < 0.075000$$

Finding the square root of each part (before rounding), then rounding to two decimal places, yields this 95% confidence interval estimate of the population standard deviation:

$$0.10 \text{ volt} < \sigma < 0.27 \text{ volt.}$$

Based on this result, we have 95% confidence that the limits of 0.10 volt and 0.27 volt contain the true value of σ.

Determining Sample Sizes

The procedures for finding the sample size necessary to estimate σ^2 are much more complex than the procedures given earlier for means and proportions. Instead of using very complicated procedures, we will use Table 7-2.

STATDISK also provides sample sizes. With STATDISK, select Analysis, Sample Size Determination, and then Estimate St Dev.

Minitab, Excel, and the TI-83/84 Plus calculator do not provide such sample sizes.

Determining Sample Sizes

Sample Size for σ^2		Sample Size for σ	
To be 95% confident that s^2 is within	of the value of σ^2, the sample size n should be at least	To be 95% confident that s is within	of the value of σ, the sample size n should be at least
1%	77,208	1%	19,205
5%	3,149	5%	768
10%	806	10%	192
20%	211	20%	48
30%	98	30%	21
40%	57	40%	12
50%	38	50%	8
To be 99% confident that s^2 is within	of the value of σ^2, the sample size n should be at least	To be 99% confident that s is within	of the value of σ, the sample size n should be at least
1%	133,449	1%	33,218
5%	5,458	5%	1,336
10%	1,402	10%	336
20%	369	20%	85
30%	172	30%	38
40%	101	40%	22
50%	68	50%	14

Example:

We want to estimate the standard deviation σ of all voltage levels in a home. We want to be 95% confident that our estimate is within 20% of the true value of σ. How large should the sample be? Assume that the population is normally distributed.

From Table 7-2, we can see that 95% confidence and an error of 20% for σ correspond to a sample of size 48. We should obtain a simple random sample of 48 voltage levels form the population of voltage levels.

Recap

In this section we have discussed:

- **The chi-square distribution.**
- **Using Table A-4.**
- **Confidence intervals for the population variance and standard deviation.**
- **Determining sample sizes.**