Introduction to Measurements and Uncertainty

Measurements in real-world situations: deviations, random errors, and systematic errors

When you make any measurement of a physical system, you will not determine the "correct" answer. That is because our measuring tools and clocks have limitations so any measurement we make is only an approximation of the "correct" answer. Not only that, but if we make repeated measurements of the same system, we will find that our answers are not always the same; they tend to fluctuate around a given value. After making multiple measurements, we compute the mean value *m*, which we take as our best approximation to the "correct" answer. For *n* measurements of quantity *x*, the mean *m* is given as

$$m = {(\sum_{i=1}^{n} x_i)}/{n} \quad (1)$$

The individual measurements will differ from the mean by various amounts. These *deviations* may be large or small, but for a given set of measurements, most of the deviations will fall in some range. If we have many measurements, we can compute the *experimental standard deviation, s*. The standard deviation gives us an estimate of how large the deviations of subsequent measurements from the mean are likely to be. In this class, we generally won't have enough measurements to compute a meaningful standard deviation, so we will estimate the range of our measurements another way (see below).

For *n* measurements of quantity *x* with mean *m*, the experimental standard deviation *s* is given as

$$s = \sqrt{{(\sum_{i=1}^{n}(x_i - m)^2)}/{n}} \quad (2)$$

Your calculator and spreadsheet have built-in functions for this! There is no need to calculate this by hand!

If we redo the whole experiment, perhaps on another day, with another set of *n* measurements, we will not find the same mean! If we repeat the whole experiment many times, we can take a grand mean of all the mean values which will generally be a better estimate of the "correct" answer than any of our individual mean values. We can also compute the *standard deviation in the mean,* s_m ,which gives us an estimate of the range in which the mean value of another set of experiments would fall, which is also an estimate of how close our mean value is to the "correct" answer. We will never do this in our class! However, if we have enough measurements to compute an experimental standard deviation, we can compute an estimate of the standard deviation in the mean, even though we have only one set of measurements.

Given the standard deviation *s*, the standard deviation in the mean s_m can be *estimated* from only one set of measurements

$$s_m = s/\sqrt{n} \quad (3)$$

The deviations from the mean in our measurements and the deviation of the mean value from the “correct” answer are the result of random errors, systematic errors, and blunders.

Random errors: the deviations resulting from random errors should be equally distributed above and below the “correct” answer. They should have **both** positive and negative signs. Random errors are generally not "mistakes." For example, if you are measuring a length with a ruler that has some thickness, you must hold your eye directly above the line indicating the answer. If your eye position wobbles a bit to the left or right, you will introduce “parallax,” but the deviations should be equally above and below the correct answer.

Systematic errors: the deviations resulting from systematic errors will usually all be above or below the correct answer. They will have **either** positive or negative signs, not both. Systematic errors affect the measurements in a particular experiment in a consistent, rather than random, fashion. For our ruler example, this might mean that you preferentially always kept your eye position either to the right or to the left of the correct position, always introducing the same sign of the deviation. Or, perhaps the ruler you were using was calibrated at 20 °C but the classroom has heated up to 25 °C, changing the length of your ruler. Systematic errors affect *all* observations, to a greater or lesser degree, but they can be subtle and hard to find.

Blunders: these are true mistakes. Perhaps you held the ruler backwards or made a calculation errors. You should probably not be discussing blunders in your lab report; you should fix them!

Precision, accuracy, uncertainty, and error

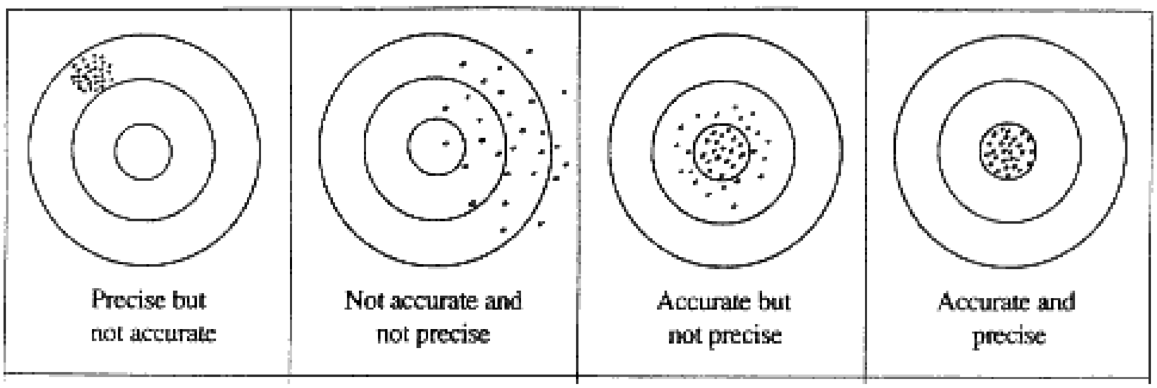
Random and systematic errors affect the accuracy and precision of your results.

The **precision** of a measurement is an indication of how finely the result has been determined, or how well we can determine an answer, not how close our answer is to the true value. Precision is related to the quality of our measuring tools, our skills in using and calibrating them, and random errors in the measuring process. It gives you an idea of how similar the result would be if you redid the experiment. When you have many measurements, the experimental standard deviation is a good estimate of the precision.

Our estimate of the precision: As we will not generally have many measurements, we will try to estimate in advance what the precision of our measurements should be. We may do this from the size of the smallest ruling on our equipment, differences in measurements made by different students, or some other criterion, as appropriate for that experiment. We will call our best estimate of the precision the *uncertainty*; uncertainty occurs in all measurements. The uncertainty in a given measurement will be denoted with the symbol σ (sigma).

The uncertainty is the range within which another measuement would be consistent with our results. A result is usually quoted with its uncertainty. For example, a velocity would be quoted with its uncertainty, $v \pm \sigma_v$, or 4.2 ± 0.1 m/s. This means that most of the time we make this measurement with the same equipment and techniques, the result would fall in the range 4.1 − 4.3 m/s.

The **accuracy** of a result is an indication of how close it is to the "true" answer. If we know the “true” answer (if, for example, it is given in the textbook), the difference between an observed value and the true value is referred to as the **error** (not to be confused with the random and systematic errors in the measurements). If we have two completely independent answers but no "textbook" answer, we will refer to the **difference**, rather than the **error**. Surprisingly, measurements with very high precision could have very low accuracy. That would likely be the result of systematic errors. Random errors can also contribute to the accuracy of a result (and not just to the precision). Measurements with poor precision are unlikely to have high accuracy, although they may, because a small number of low-precision measurements are unlikely to average to the "correct" answer, even if the systematic errors are small. The figure below illustrates the contributions of both random and systematic errors in an idealized way. We may not know the true value, but we try to minimize our errors to get as close to the true value as possible.



Summary of important terms *as used in this class*

Deviation: the difference between a given measurement and the mean value

Random error: deviations which are not preferentially positive or negative compared to the correct answer.

Systematic error: deviations which are preferentially either positive or negative compared to the correct answer.

Precision: how finely the result has been determined (usually the result of random error).

Accuracy: how close the result is to the "correct" answer (often affected by systematic *and* random errors).

Uncertainty: the range in which we expect either the individual measurements or the final computed result to fall. This is denoted with the symbol σ.

Error: The difference between an experimental result and the "correct" answer.

Propagation of errors

The discussion above refers to the precision and accuracy of individual parameters. Usually, we have to use these measurements to derive some other result. Last semester you may have computed the density of an aluminum cylinder after measuring the height, diameter, and mass. The uncertainty (precision) of the final result depends on the uncertainty of each of the measured quantities. If you did this experiment, you used an approximate method to calculate the uncertainty in the density based on the uncertainties in all of the measured quantities: height, diameter, and mass. This is called *propagation of errors.* The formula you used gave an overestimate of the uncertainty, as it assumed that each quantity would be measured with the maximum possible deviation. Generally, we use a formula which assumes that will not be the case.

When a derived value depends on measured values with uncertainties, it is necessary to use the appropriate propagation of error formula to derive the uncertainty in the derived value. These formulae are different depending on whether you are adding (or subtracting) two quantities or multiplying (or dividing) them (or applying other functional relationships).

Consider an experiment involving the measured variables A and B; A = 10. ± 1. and B = 1000. ± 10. Although A is more precisely measured than B in absolute terms ($\sigma_A < \sigma_B$) it has a greater *percentage* uncertainty (10% vs. 1%). Which quantity (σ_A or σ_B) dominates the final uncertainty here depends on whether we are adding/subtracting or multiplying/dividing A and B.

For addition and subtraction, C = A + B (or C = A – B), $\sigma_C = \sqrt{\sigma_A^2 + \sigma_B^2}$ (remember, the Greek letter sigma, σ, denotes the uncertainty or precision). You don't simply add the uncertainties because it is improbable that all measurements will be at their maximum uncertainty. With the given values, the result is C = 1010 and σ_C = 10. (rounded from 10.05) or 1%, reflecting the precision of B, the value with the larger uncertainty. The smaller uncertainty in A is simply lost in the uncertainty of B.

For multiplication and division, the uncertainty of the result depends on the percentage uncertainty, rather than the magnitude of the uncertainty, because an uncertainty in one quantity is multiplied by the other quantity. Consider if A were measured to be 11. Then C = A × B = 11,000, differing from the correct value by 10% despite the fact that B is good to 1%! Multiplying a well determined number by a poorly determined number gives you a poorly determined result.

For C = A × B (or C=A/B), $\sigma_C = C \times \sqrt{\frac{\sigma_A^2}{A^2} + \frac{\sigma_B^2}{B^2}}$. With the given values, the result is C = 10000. and σ_C = 1000. (rounded from 1005), so the resulting uncertainty is 10%, despite the higher relative precision of B.

Bringing these ideas into the lab report

Generally, we will not have enough measurements (and deviations) to compute *s*, the standard deviation. Thus, when making measurements, you should always record an estimate of the uncertainty in the measurement. Many students will choose either the smallest interval on the measuring equipment or some fraction thereof, if it is possible to interpolate within the interval. This is often a reasonable thing to do, but it is often optimistic. There are often reasons why you cannot measure as finely as the rulings on your equipment. It is the job of each student to assess this as realistically as possible. Note: it is *very* unlikely that you can interpolate to 1/10 of a division. **In all lab reports, you must discuss your choice of the uncertainty in each measurement (as well as sources of uncertainty that you could not estimate numerically).**

In our labs, we frequently want to compare our results with the "correct" answer from the textbook or other reference work. The difference between your result and the "correct" answer is called the *error*. If you have an estimated uncertainty, you should ***ALWAYS*** compare the error to the size of the uncertainty. The size of the uncertainty is an estimate of the range in which our answers would fall if we did the experiment again.

Depending on the experiment, the uncertainty may simply be your estimated uncertainty, as described above, or a standard deviation. In the "error analysis" section of the lab writeup, you may be given a propagation of errors formula appropriate for that lab, or you may be asked to assess the uncertainty of the result with a graphical procedure. If the experiment has been done without significant systematic errors, and if the uncertainty has been estimated correctly, then the error should be smaller than the uncertainty (remember, the error is difference between your result and the "correct" answer). The error may be large, but if it is within the uncertainty, we have determined the answer as well as we can with the equipment and techniques available. If the error is larger than the uncertainty, we have not done as well as we should have, and it is your job to understand why! For example, if the expected uncertainty in a projectile launch is 0.1 m/s, any result within 0.1 m/s of the predicted answer would be acceptable. On the other hand, if the answer differs by 0.3 m/s from the predicted answer, more than the expected uncertainty, that might be an indication that systematic errors were important. Alternatively, the estimate of the uncertainty may have been overly optimistic.

Many students are used to reporting a "percent error." This is the percent by which your experimental result differs from the "correct" answer. You might consider that a 2% error is good while a 20% error is bad, but that is not necessarily so. If your equipment was good enough that your uncertainty is 1%, then a 2% error is not good! If our equipment was only good enough to expect an uncertainty of 40%, then a 20% error is acceptable. That is why we always want to compare the error to the uncertainty, rather than using percent errors.

However, if you also wish to present a percent error, you may do so. If you are comparing your result to a "correct" answer you looked up (the "theoretical" result), use a percent error. If there is no "correct" answer, and you are comparing two of your own experimental results, use a percent difference.

Percent error

$$\%error = \frac{\left|(experimental\ result - theoretical\ result)\right|}{theoretical\ result} \times 100\%$$

Percent difference

$$\%difference = \frac{\left|(result_1 - result_2)\right|}{average} \times 100\%$$

References

Probability and Experimental Errors in Science, Lyman G. Parratt

Data Reduction and Error Analysis for the Physical Sciences, Philip R. Bevington and D. Keith Robinson

Significant Figures and Uncertainty

When doing computations, your calculator will give you many digits, but not all of these have any real meaning. Which of the decimal places have meaning is related to the size of your uncertainties. These are known as significant figures (or significant digits). In your lab report, numbers should be given with the appropriate number of significant figures, both for the measured values and for the computed values.

When recording your measurements, you should use the correct number of significant figures. That is determined by the uncertainty you have estimated for that measurement. Uncertainties are added or subtracted to your result, so the addition/subtraction rule for significant digits applies. For example, if your estimated uncertainty is 0.1 units, then you should record your measurement to one digit past the decimal place (or two), for example 12.2 units or 12.25 units. If the last decimal place is a zero, keep it! For example, with the uncertainty shown above, 12.0 units would be the correct way to record the result, but 12 units would not.

When doing computations, there are two rules for significant figures. For multiplication or division, retain the same number of significant figures as in the number with the fewest significant figures. For addition or subtraction of two numbers, you keep the least significant digit common to both numbers. In this class, you can always keep an extra digit. Also, don't round off each step in a calculation to this small number of significant figures. Keep extra digits in your calculator and round off at the end.

When computing the uncertainty in your final results (when required by the instructions for the lab), there is no reason to keep more than one significant digit (or at most two) in your estimated uncertainty. The uncertainty tells you the range within which your answer may be expected to fall; to say that your uncertainty is ±0.00012345 is no more informative than it is to say that your uncertainty is ±0.0001 or ±0.00012, and it's a lot messier. That also determines how many digits you should keep in your result. Uncertainties are added or subtracted. For addition or subtraction of two numbers, you keep the least significant digit common to both numbers. Thus, 1.234567 ± 0.00321 should be given as 1.234 ± 0.003

Finally, all results and their uncertainties should be quoted with the same exponent, for ease of comparison, as should all related numbers (see the section below on your summary table).

Guidelines for Laboratory Experiments and Notebooks

What you do in class

- Come to class on time. If you are late for lab, you may have points deducted or be required to leave.
- Treat the experiments as if they were real research.
- Your grade can include how carefully you perform your work.
- A laboratory notebook is a **complete** record of your observations, measurements, and calculations.
- **Record all data yourself. Do not rely on your lab partners to record the data and copy it from them later! You will lose points!**
- **The lab notebook should be the <u>original</u> record. Write everything directly into your lab notebook. <u>Do not</u> record data on the Laboratory Description or scratch paper and copy it over later into your lab notebook! You will lose points!**
- **Record all original data in ink.**
- The data you record and the results you calculate are not "just numbers." They represent real physical quantities. When you record data or do a calculation, think about whether the number is plausible. Use common sense.
- For each measurement you make, try to estimate an uncertainty. Record the uncertainty and state how you determined it (see the earlier discussion of measurements and uncertainty).
- All measurements should be recorded, even preliminary trials and mistakes. If you make a mistake, cross out the data and note what was wrong. **Do not erase!**
- **Clarity** is important. Leave space between lines.
- To the extent possible, **record data in tables** with labeled columns (and units). After performing a few trial measurements to make sure you have identified all the required variables, prepare a table in which to record your data, preferably with a ruler.
- Make a simple **diagram** of the setup. Include it with the lab report. Don't copy diagrams from the lab manual. **Carefully label each measured quantity** that will be used in a calculation. The diagram should be large enough to clearly show what has been measured.
- Neatness counts! You will lose points for messy or disorganized reports. Use rulers for all straight lines.
- **Ask your instructor to initial your data before you leave**

Laboratory Reports

What you turn in for a grade

Some labs may require a different format. The point distribution depends on the specifics of each lab; a possible point distribution is shown. You may need to vary the format for a specific lab.
Do not turn in the printed pages from the lab description or manual.
There are penalties for late or very messy lab reports!

Title, your name, your partner's name, date of the lab and report **on the 1st page**		–1
Summary Table (page 1)	• List the major results of each part of the experiment. Your instructor will give you a specific format for each lab. In addition to the experimental results, you will often have to indicate experimental uncertainty and an error derived by comparing your experimental result to a calculated value or an alternate experimental result.	2
Conclusions (on page 1, below the summary table) **This portion of the lab report must be typed.**	• Briefly summarize the experiment. What did you do? What did you determine? Do **not** include the *details* of your procedure in the conclusions**. THIS IS NOT A "PURPOSE" SECTION.** • **Explain** the physical principles that are the basis for the experiment. Sets of equations are **not** explanations (occasionally not required). • What were your results? Summarize the main numerical results (not data − results)!!! • **Were the differences between your results and predicted results consistent with your estimates of the expected uncertainty (due to random errors)?** • Discuss specific sources of experimental errors – "computation error" and "human error" are not acceptable! Comment on the relative importance of random vs. systematic errors.	3
Sample Calculations, Graphs, Diagrams.	• Include sample calculations and graphs, if applicable. • At least one sample calculation must be shown for each separate step in the calculation. **This should be on a separate page or pages, NOT the summary page or data page!** • For each calculation, first give the equation using symbols. Then give the equations with numbers and **units**. • Graphs should use an entire page. Axes should be labeled. Each axis should have a scale, including units. If you plot the points by hand, use a ruler to draw the axes. • Include a diagram for each experiment **with each measured variable clearly labeled**. The diagrams do not need to be artistic, but should be clear enough so a reader can tell what you were measuring.	3
Data **Your original data go at the END of your report.**	• **Attach the signed pages from your lab notebook with the ORIGINAL, signed data. Do NOT copy over from scratch paper!** • **DATA MUST BE IN PEN!** • **Data should be recorded in a neat and organized format. It should be organized in horizontal rows and vertical columns.** Labs with disorganized data may be given reduced or no credit.	2 0 if copied 
Total		10

The First Page of Your Lab Report

Summary Table

The very first portion of your lab report will be the summary table. This is a table of results, not data. The specific format of each summary table will be given in the lab writeup for each experiment. For large or small numbers, scientific notation should be used, with one modification. All related numbers should be written with the same exponent. For example, if your measurement is 1.23×10^6 and the uncertainty is 1 unit in the second decimal place, this number should be written as $(1.23 \pm 0.01) \times 10^6$ NOT $1.23 \times 10^6 \pm 1 \times 10^4$, even though that appears to violate the rules about scientific notation. Even better, put the exponent in the column heading.

Sample Conclusions

The conclusions are a very important part of a lab report. Here is a sample conclusion, appropriate for the projectile motion lab, with notations on required elements:

We used a spring loaded projectile launcher to launch a steel ball horizontally and at an angle from the laboratory tables. We calculated the velocity from the horizontal launch, and we used that velocity to predict the range when fired at an angle. We then tested our predictions experimentally.

Summarize the experiment. *Note important details such as special equipment used, but do not restate each step of the procedure.*

Because velocity and acceleration are vectors, and gravitational acceleration is in the vertical direction, we were able to separate the x and y components and use the equations of projectile motion in two dimensions to calculate the initial velocity and predicted range.

Briefly explain the "physics" behind the experiment.

We determined the initial velocity was 4.23 ± 0.1 m/s. We predicted that the range would be 3.55 ± 0.1 m when the ball was launched at an angle of 33º. Our actual measured range was 3.85 m. The difference between our predicted and measured values was 0.30 m (or 8.5%).

Summarize the final results numerically. These are RESULTS NOT DATA! Compare your numbers with established results or your predictions.

The difference between our predicted and measured values is greater than we expected based on the estimated uncertainty. The random error in the initial velocity was derived from the actual range in launch distances in the initial part of the experiment, approximately ± 0.05 m. Other sources of uncertainty in the experiment include uncertainties in the measured height of the table (±0.01 m) and angle of launch (± 1º). These uncertainties were one division of the ruler and protractor, respectively. Because of the high density and low speed of the ball, it is unlikely that air resistance has any significant effect. This suggests that some other, undetermined, systematic error is important.

Are the results in the expected range?

How do you account for the differences? What are the specific sources of uncertainty? How did you determine the sizes of the uncertainties?