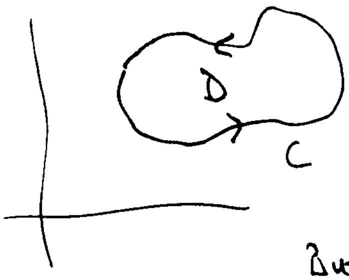
## Notes on Green's Theorem (Ideas related to §16.4, but not in the textbook.)



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

But $C$ must be a closed curve.

Remark: You can extract information $\int_{C_1} P\,dx + Q\,dy$ even when $C_1$ is not closed

$C_1$ = not closed

$C_2$ = designed to enclose a region $D$ so that $C_1 + C_2 = C$ where $C$ = boundary of $D$

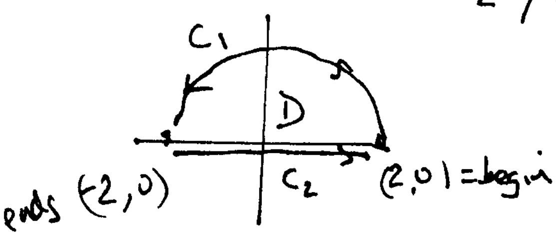
Then

$$\underbrace{\int_{C_1} P\,dx + Q\,dy}_{\text{Maybe this is ugly}} + \underbrace{\int_{C_2} P\,dx + Q\,dy}_{\text{easy}} = \underbrace{\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA}_{\text{easy}}$$

So $$\int_{C_1} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA - \int_{C_2} P\,dx + Q\,dy$$

ex: $\displaystyle\int_{C_1} (xe^{-x^2/2} - y)\,dx + (ye^{-y^2/2} + x)\,dy$

where $C_1 : \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ $\quad 0 \le t \le \pi$




Q: Is C closed? No

Q: Is $\langle xe^{-x^2/2} - y,\ ye^{-y^2/2} + x \rangle$ conservative?

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}(ye^{-y^2/2} + x) - \frac{\partial}{\partial y}(xe^{-x^2/2} - y)$$

$$= 1 - (-1) = 2 \ne 0$$

No, $\langle P, Q\rangle$ is not conservative but $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 2$ is a very simple function of two variables

Hmmm. It looks like we must calculate $\displaystyle\int_{C_1} P\,dx + Q\,dy$ directly.

But it gives rise to a horrible integral.

Let's a design a curve $C_2$ to be the segment of the x-axis from $(-2,0)$ to $(2,0)$

There $\begin{cases} x = x \\ y = 0 \end{cases}$ $\quad dx = 1\,dx$, $\quad dy = 0$

$-2 \le x \le 2$

because $-x\,dx = d(-\frac{1}{2}x^2)$

$$\int_{C_2} (xe^{-x^2/2} - \cancel{y}^{\,0})\,dx + (ye^{-y^2/2} + x)\,\cancel{dy}^{\,0} = \int_{-2}^{2} xe^{-x^2/2}\,dx = -\int_{-2}^{2} e^{-x^2/2}\,d(-\tfrac{x^2}{2})$$

$$= -\left[e^{-x^2/2}\right]_{-2}^{2} = -e^{-2} + e^{-2} = 0$$

Also, this integral should be easy:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D 2\, dA = 2 \iint_D dA$$

$$= 2 \{ \text{area of half-disk of radius } 2 \}$$

$$= 2\left(\tfrac{1}{2}\pi 2^2\right) = 4\pi$$

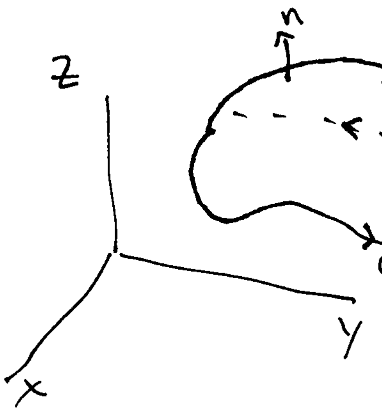
ugly integral $= \int_{C_1} P\,dx + Q\,dy = \int_{C_1} (x e^{-x^2/2} - y)\,dx + (y e^{-y^2/2} + x)\,dy$

where $C_1$ = semicircle

Answer $= \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA - \int_{C_2} P\,dx + Q\,dy$

$= \quad 4\pi \quad - \quad 0 \quad = 4\pi$

# 16.8 Stoke's' Theorem




Let $S$ be a piecewise-smooth, oriented surface that is bounded by a simple closed curve $C$, with positive orientation (right-hand rule)

Let $\vec{F}$ be a vector field [blah blah]

Then $$\boxed{\int_C \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\,\vec{F}\cdot d\vec{S}}$$

or $$\int_C \vec{F}\cdot\vec{T}\,ds = \iint_S \text{curl}\,\vec{F}\cdot\vec{n}\,dS$$

Remark: If $S$ is a subset of the xy-plane, so $\vec{n}=\vec{k}$,

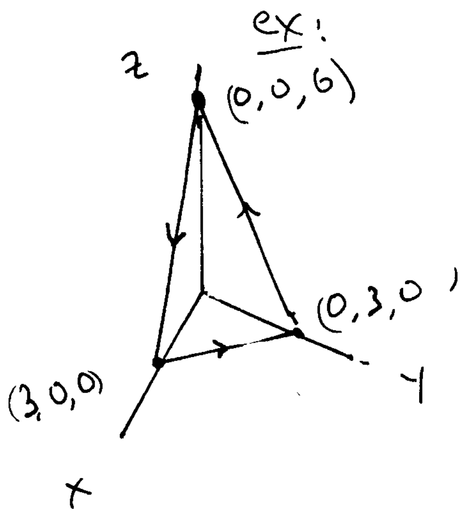
and $\vec{F}=\langle P, Q, 0\rangle$ then

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & 0 \end{vmatrix} = (\text{something})\,\hat{i} + (\text{something})\,\hat{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

So that $$\text{curl}\,\vec{F}\cdot\vec{n} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}\cdot\vec{k} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

thus $$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy = \iint_S \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

That is, Green's Theorem is a special case of Stokes' theorem.




$S$ is the portion of the plane in the first octant with intercepts $(3,0,0)$, $(0,3,0)$, $(0,0,6)$.

That is $\frac{x}{3} + \frac{y}{3} + \frac{z}{6} = 1$

or $2x + 2y + z = 6$,

with upward normal.

Also, suppose $C$ is the triangular boundary of $S$ with positive orientation.

Find $\int_C \vec{F} \cdot d\hat{r}$ where $\vec{F}(x,y,z) = \langle -y^2, z, x \rangle$

by <u>instead</u> using Stokes' Theorem and calculate a surface integral.

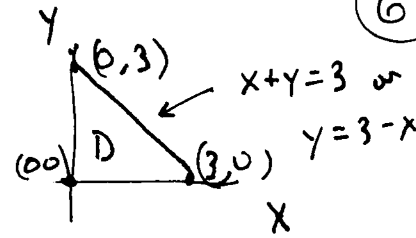
$$\text{curl } \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & z & x \end{vmatrix} = \left( \frac{\partial x}{\partial y} - \frac{\partial z}{\partial z} \right) \hat{i} + \left( \frac{\partial(-y^2)}{\partial z} - \frac{\partial x}{\partial x} \right) \hat{j} + \left( \frac{\partial z}{\partial x} - \frac{\partial(-y^2)}{\partial y} \right) \hat{k}$$

$$= -\hat{i} - \hat{j} + 2y\hat{k} = \langle -1, -1, 2y \rangle$$

Next set up $\iint_S \text{curl } \vec{F} \cdot d\vec{S} = \iint_D \text{curl } \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) \, du\, dv$

But, for convenience, we'll let $(x,y)$ be the parameters for the plane $S$.

$$S: \begin{cases} x(x,y) = x \\ y(x,y) = y \\ z(x,y) = 6-2x-2y \end{cases}$$




That is $\quad \vec{r}(x,y) = \langle \underset{\hat{i}}{x}, \underset{\hat{j}}{y}, \underset{\hat{k}}{6-2x-2y} \rangle$

$$\vec{r}_x = \langle 1, 0, -2 \rangle$$

$$\vec{r}_y = \langle 0, 1, -2 \rangle$$

$$(\vec{r}_x \times \vec{r}_y) = \langle 0-(-2), 0-(-2), 1-0 \rangle$$

$$= \langle 2, 2, 1 \rangle$$

$$\iint_D \operatorname{curl} \vec{F}(\vec{r}(x,y)) \cdot (\vec{r}_x \times \vec{r}_y)\, dx\,dy$$

$$= \int_0^3 \int_0^{3-x} \langle -1, -1, 2y \rangle \cdot \langle 2, 2, 1 \rangle \, dy\, dx$$

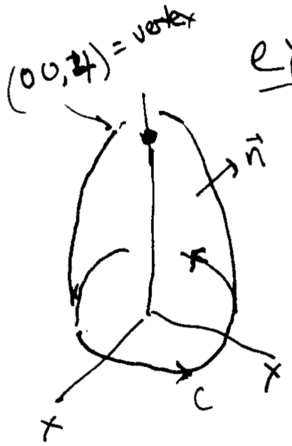
$$= \int_0^3 \int_0^{3-x} (-2-2+2y)\, dy\, dx = \int_0^3 \int_0^{3-x} (2y-4)\, dy\, dx$$

$$= \int_0^3 \left[ y^2 - 4y \right]_0^{3-x} dx = \int_0^3 \left[ (3-x)^2 - 4(3-x) \right] dx$$

$$= \int_0^3 (3-x)\,[3-x-4]\, dx \int_0^3 (x-3)(x+1)\, dx$$

$$= \int_0^3 (x^2-2x-3)\, dx = \left[ \frac{x^3}{3} - x^2 - 3x \right]_0^3 = 9-9-9$$

$$= \boxed{-9}$$




ex: For the surface $S: \begin{cases} z = 4 - x^2 - y^2 \\ z \geq 0 \end{cases}$

So intersection with xy-plane is $0 = 4 - x^2 - y^2$

or $x^2 + y^2 = 4$

Suppose

$$F(x,y,z) = \langle 2z, \ x, \ y^2 \rangle$$

Find $\iint_S \text{curl}\, \vec{F} \cdot d\vec{S}$ by instead using Stokes' Theorem to instead calculate a line integral:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,\cancel{dz}^{\,0}$$

$\to 0$ because C lies in the plane $z = 0$.

$C: \begin{cases} x = 2\cos t \\ y = 2\sin t \\ z = 0 \\ 0 \leq t \leq 2\pi \end{cases}$

$$= \int_C 2\cancel{z}^{\,0}\,dx + x\,dy + y^2\,\cancel{dz}^{\,0}$$

$$= \int_C x\,dy = \int_0^{2\pi} (2\cos t)(2\cos t)\,dt$$

$dy = 2\cos t\,dt$

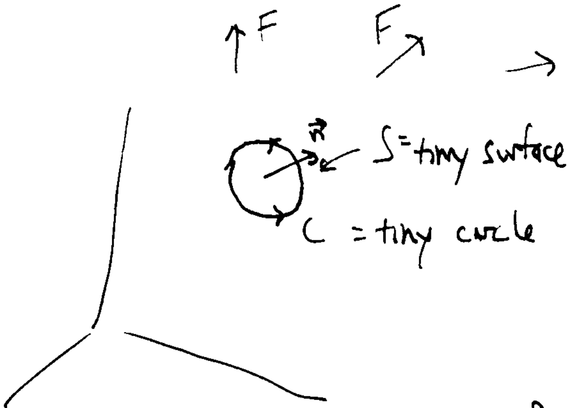
$$= \int_0^{2\pi} 4\cos^2 t\,dt = \int_0^{2\pi} 4 \cdot \frac{(1 + \cos 2t)}{2}\,dt$$

$$= \int_0^{2\pi} (2 + 2\cos 2t)\,dt$$

$$= 2\int_0^{2\pi} dt + 2\cancel{\int_0^{2\pi} \cos 2t\,dt}^{\,0} = 2(2\pi - 0)$$

$$= 4\pi$$

## Physical interpretation of curl $\vec{F}$




Assume curl $F \simeq$ constant on $S$, where $S$ is so small that it is a planar disk of radius $a$, with unit normal $\vec{n}$, so that curl $\vec{F} \cdot \vec{n} \approx$ constant.

By Stokes' Theorem,

$$\int_C \vec{F} \cdot \vec{T}\, ds = \iint_S \operatorname{curl} \vec{F} \cdot \vec{n}\, dS = (\operatorname{curl} F \cdot \vec{n}) \iint_S dS = (\operatorname{curl} \vec{F} \cdot \vec{n}) \cdot (\text{area of } S)$$

$$= (\operatorname{curl} \vec{F} \cdot \vec{n})(\pi a^2)$$

So that

$$\boxed{\operatorname{curl} \vec{F} \cdot \vec{n} \approx \frac{1}{\pi a^2} \int_C \vec{F} \cdot \vec{T}\, ds}$$

, where $a$ is a small radius.

The physical interpretation of this is that if $\vec{F}$ represents fluid flow, curl $\vec{F} \cdot \vec{n}$ represents the tendency of a paddlewheel, whose axis is aligned with the unit vector $\vec{n}$, to rotate counterclockwise about $\vec{n}$. The direction of the vector curl $\vec{F}$ is that choice of vector $\vec{n}$ which maximizes this counterclockwise rotational tendency.