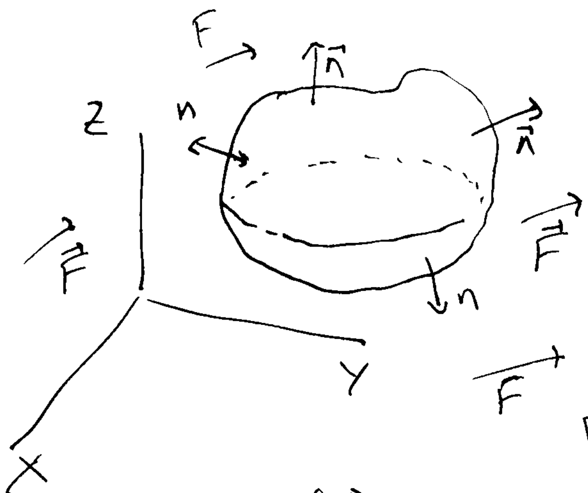
Math 252

17 May 2018

16.9 Divergence Theorem E = "simple" solid, S = its (closed) boundary

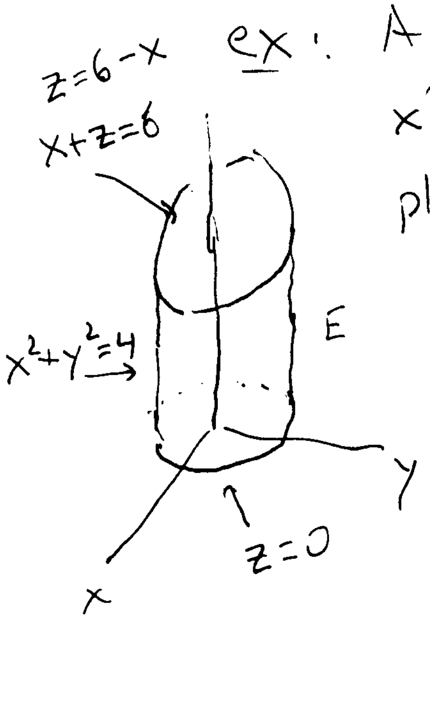


$\vec{n}$ = outward unit normal

$\vec{F}(x,y,z)$ = vector field in space

Divergence Theorem:

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \boxed{\iint_S \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\,dV}$$



ex: A solid E is bounded by the cylinder $x^2+y^2=4$, by the xy-plane, and by the plane $x+z=6$. Let S be the boundary of E.

Find $\displaystyle\iint_S \vec{F}\cdot\vec{n}\,dS$

where $\vec{F}(x,y,z) = \langle x^2+\sin z,\ xy+\cos z,\ e^y\rangle$

We'll calculate this by using the triple integral side of the Divergence Theorem.

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial R}{\partial z} + \frac{\partial Q}{\partial y} + \frac{\partial P}{\partial x}$$

$$= \vec{\nabla} \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}\left[x^2 + \sin z\right] + \frac{\partial}{\partial y}\left[xy + \cos z\right] + \frac{\partial}{\partial z}\left[e^y\right]$$

$$= 2x + x + 0 = 3x$$

$$\iiint_E \operatorname{div} F \, dV = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{6-x} 3x \, dz\, dy\, dx \leftarrow$$ rectangular coordinates (not recommended)

Boundaries

rectangular		cylindrical
$z = 6 - x$	$\Rightarrow$	$z = 6 - r\cos\theta$
$x^2 + y^2 = 4$	$\Rightarrow$	$r = 2$
$z = 0$	$\Rightarrow$	$z = 0$

$$= \int_0^{2\pi} \int_0^{2} \int_0^{6 - r\cos\theta} (3r\cos\theta)\, r \, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} 3r^2\cos\theta \left[z\right]_0^{6-r\cos\theta} dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} (3r^2\cos\theta)(6 - r\cos\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{2} (18r^2\cos\theta - 3r^3\cos^2\theta)\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[6r^3\cos\theta - \frac{3}{4} r^4\cos^2\theta\right]_0^2 d\theta$$

$$= \int_0^{2\pi} (48\cos\theta - 12\cos^2\theta)\, d\theta$$

$$= \int_0^{2\pi} \left(48\cos\theta - 6 - 6\cos 2\theta\right) d\theta$$

Using: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

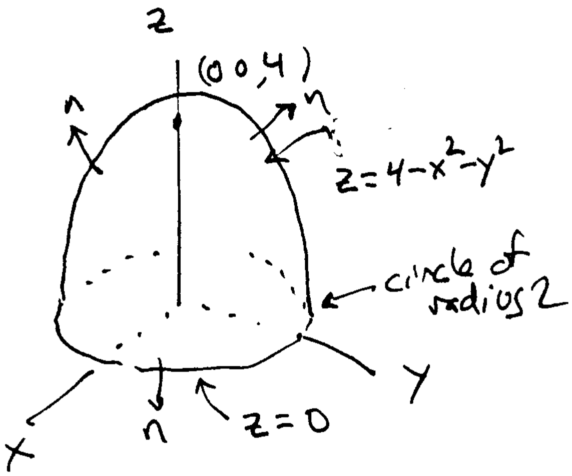
$$= \Big[48\sin\theta - 6\theta - 3\sin 2\theta\Big]_0^{2\pi}$$

$$= -6(2\pi) = -12\pi$$

16.9 (cont'd) Divergence Theorem

Given $\iint_S \vec{F}\cdot\vec{n}\,dS$		Is $\vec{F}$ a curl? No	Is $\vec{F}$ a curl? Yes
Is S a closed surface	No	Calculate directly	Stokes theorem (do a line integral instead)
	YES	Divergence theorem (do a triple integral instead)	0

ex: Let E be the solid region between the paraboloid $z=4-x^2-y^2$ and the xy-plane, so that its boundary is the (piecewise-smooth) surface S.



Task: For the vector field

$$\vec{F}(x,y,z)=\langle 2z, x, y^2\rangle$$

evaluate $\iint_S \vec{F}(x,y,z)\cdot\vec{n}\,dS$.

Can we use the Divergence Theorem? Yes, as S is a closed surface.

ex (cont'd)

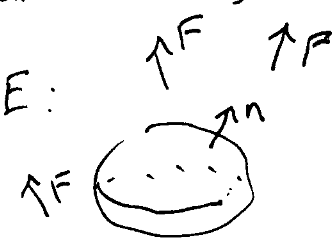
$$\operatorname{div} F(x,y,z) = \operatorname{div} \langle 2z, x, y^2 \rangle$$
$$= \frac{\partial}{\partial x}[2z] + \frac{\partial}{\partial y}[x] + \frac{\partial}{\partial z}[y^2]$$
$$= 0 + 0 + 0 = 0 \quad \text{(Whoa.)}$$

Apply the Div. Thm:

$$\iint_S F(x,y,z)\cdot\vec{n}\,dS = \iiint_E (\operatorname{div} F)\,dV = \iiint_E 0\,dV = \boxed{0}$$

<u>Physical interpretation</u> of div $F(x,y,z)$

Imagine a tiny sphere E:

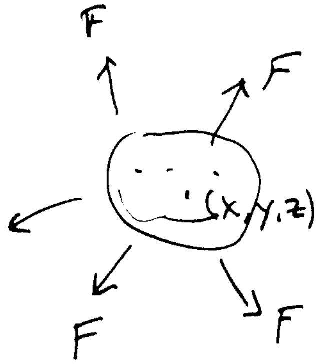


with boundary S and outward unit normal $\vec{n}$.

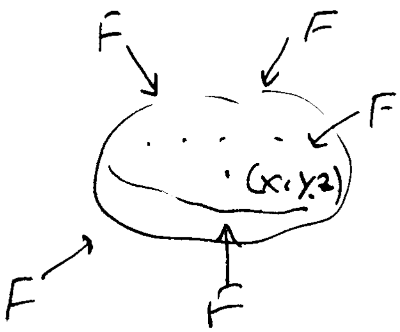
The sphere is so tiny that, for a vector field $\vec{F}(x,y,z)$, div $\vec{F}(x,y,z)$ is essentially constant.

$$\iint_S \vec{F}(x,y,z)\cdot\vec{n}\,dS = \iiint_E \operatorname{div} F(x,y,z)\,dV$$
$$= \operatorname{div} F(x,y,z) \underbrace{\iiint_E dV}_{\text{volume}(E)}$$

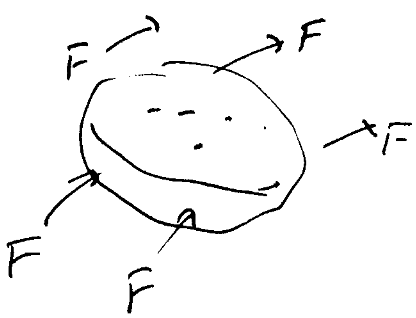
$$\Rightarrow \boxed{\operatorname{div} F(x,y,z) = \frac{1}{\operatorname{vol}(E)} \iint_S \vec{F}\cdot\vec{n}\,dS}$$



div $F > 0$ $\Rightarrow$ vector field $\vec{F}$ has a <u>source</u> at (x,y,z)



div $F < 0$ $\Rightarrow$ $\vec{F}$ has a <u>sink</u> at (x,y,z)

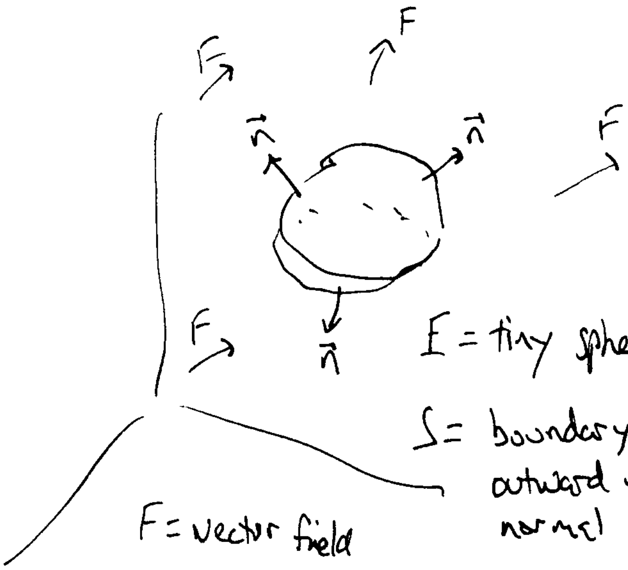


div $F = 0$ $\Rightarrow$ $\vec{F}$ is <u>incompressible</u> at (x,y,z).

<u>Fact</u> from physics; If $\vec{B}$ is a magnetic field,

div $\vec{B} = 0$ by one of Maxwell's equations.

Physical Interpretation of $\operatorname{div} \vec{F}$



E = tiny sphere

S = boundary with outward unit normal $\vec{n}$

F = vector field

Assume the solid E is so tiny that $\operatorname{div} F$ is essentially constant on E.

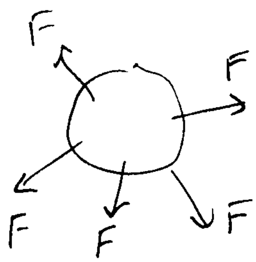
$$\iint_S \vec{F}\cdot\vec{n}\, dS = \iiint_E \operatorname{div} F\, dV$$

$$= (\operatorname{div} F) \iiint_E dV$$

$$= (\operatorname{div} F)\,(\text{volume}(E))$$

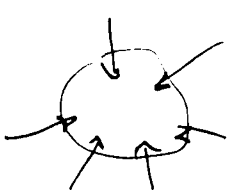
$$\operatorname{div} F \approx \frac{1}{\text{vol}(E)} \iint_S \vec{F}\cdot\vec{n}\, dS$$, where E is very small

Source

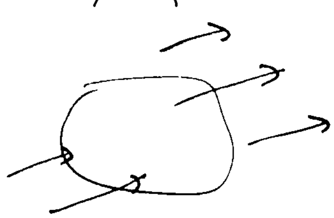


$\operatorname{div} F$ is positive

Sink



$\operatorname{div} F$ is negative



$\operatorname{div} F = 0$

"incompressible flow"