12.3 Dot Product (cont'd)

<u>Properties</u>:

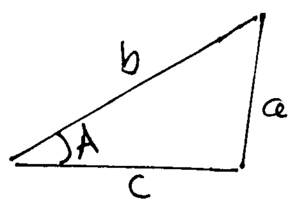
1. $\vec{a}\cdot\vec{a} = |\vec{a}|^2$
2. $\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a}$ commutative property
3. $\vec{a}\cdot(\vec{b}+\vec{c}) = \vec{a}\cdot\vec{b} + \vec{a}\cdot\vec{c}$ distributive property
4. $(c\vec{a})\cdot\vec{b} = c(\vec{a}\cdot\vec{b}) = \vec{a}\cdot c\vec{b}$ associative property

<u>Theorem</u>: If θ is the angle between $\vec{a}$ and $\vec{b}$, then

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta .$$

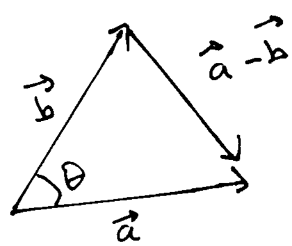
<u>Sketch of proof</u>: [Based on the Law of Cosines]

Recall the Law of Cosines



$$a^2 = b^2 + c^2 - 2bc\cos A$$

$$|\vec{a}-\vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2|\vec{a}|\,|\vec{b}|\cos\theta$$



$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|\vec{a}|\,|\vec{b}|\cos\theta$$

By property 1

$$\vec{a}\cdot\vec{a} - 2\vec{a}\cdot\vec{b} + \vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{a} + \vec{b}\cdot\vec{b} - 2|\vec{a}|\,|\vec{b}|\cos\theta$$

$$\cancel{|\vec{a}|^2} - 2\vec{a}\cdot\vec{b} + \cancel{|\vec{b}|^2} = \cancel{|\vec{a}|^2} + \cancel{|\vec{b}|^2} - 2|\vec{a}|\,|\vec{b}|\cos\theta$$

$$-2\vec{a}\cdot\vec{b} = -2|\vec{a}|\,|\vec{b}|\cos\theta \qquad \text{Divide by } -2:$$

$$\vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

<u>Corollary</u>: If $\vec a \neq \vec 0$ and $\vec b \neq \vec 0$ then

$$\cos\theta = \frac{\vec a \cdot \vec b}{|\vec a|\,|\vec b|}$$

<u>examples</u>: Recall $\vec i = \langle 1,0,0\rangle$ $\vec j = \langle 0,1,0\rangle$ $\vec k = \langle 0,0,1\rangle$

$$\vec i\cdot\vec i = \langle 1,0,0\rangle\cdot\langle 1,0,0\rangle = 1^2+0^2+0^2 = 1$$

$$\vec i\cdot\vec k = \langle 1,0,0\rangle\cdot\langle 0,0,1\rangle = (1)(0)+0^2+(0)(1) = 0$$

<u>Facts</u>
$$\left\{\begin{array}{l} \vec i\cdot\vec i = |\vec i|^2 = 1 = \vec j\cdot\vec j = \vec k\cdot\vec k \\ \vec i\cdot\vec j = \vec i\cdot\vec k = \vec j\cdot\vec k = 0 \end{array}\right.$$

<u>Vocabulary</u>: $\vec i, \vec j$ and $\vec k$ are "orthogonal, unit vectors"

<u>Remark</u>: Given the above facts, together with the properties of dot product make for an equivalent definition of dot product.

<u>example</u>: Find $\langle 3,0,-4\rangle\cdot\langle 2,10,5\rangle$ using this remark.

$$(3\vec i - 4\vec k)\cdot(2\vec i + 10\vec j + 5\vec k)$$

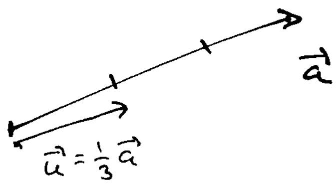
$$= (3\vec i - 4\vec k)\cdot(2\vec i) + (3\vec i - 4\vec k)\cdot 10\vec j + (3\vec i - 4\vec k)\cdot(5\vec k)$$

$$= (3)(2)\cancelto{1}{\vec i\cdot\vec i} - (4)(2)\cancelto{0}{\vec i\cdot\vec k} + (3)(10)\cancelto{0}{\vec i\cdot\vec k} - (4)(10)\cancelto{0}{\vec j\cdot\vec k}$$
$$+ (3)(5)\cancelto{0}{\vec i\cdot\vec k} - (4)(5)\cancelto{1}{\vec k\cdot\vec k}$$

$$= (3)(2)(1) - (4)(5)(1) = (3)(2) + (-4)(5) = 6 - 20 = -14$$

ex: Find a vector of length 1 in the direction of $\vec{a} = \langle 2, 1, -2 \rangle$,

$$|\vec{a}| = \sqrt{2^2 + 1^2 + (-2)^2} = \sqrt{4+1+4} = \sqrt{9} = 3$$



Let $\vec{u} = \frac{1}{3}\vec{a}$

$$= \frac{1}{3}\langle 2, +1, -2 \rangle$$

$$= \left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle$$

Is the length of $\vec{u}$ equal to 1?

$$|\vec{u}| = \sqrt{\left(\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 + \left(\frac{-2}{3}\right)^2}$$

$$= \sqrt{\left(\frac{1}{3}\right)^2 \left[2^2 + 1^2 + (-2)^2\right]}$$

$$= \sqrt{\left(\frac{1}{3}\right)^2}\ \sqrt{2^2 + 1^2 + (-2)^2}$$

$$= \frac{1}{|a|} \cdot |\vec{a}| = \left(\frac{1}{3}\right)(3) = 1$$

Fact: If c is a scalar and $\vec{a}$ is a vector, then

$$|c\vec{a}| = \underset{\text{absolute value}}{|c|}\ \underset{\text{length}}{|\vec{a}|}.$$

ex: Find a vector $\vec{b}$ of length 9 in the direction of $\vec{a} = \langle 2, 1, -2 \rangle$.

answer: Let $\vec{b} = 9\vec{u} = (9)\left(\frac{1}{3}\right)\vec{a} = 9\left\langle \frac{2}{3}, \frac{1}{3}, \frac{-2}{3} \right\rangle = \langle 6, 3, -6 \rangle$.

12.3 20) Find the angle between

$\vec{a} = \langle 1, 2, -2 \rangle$ and

$\vec{b} = \langle 4, 0, -3 \rangle$. Use $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|\,|\vec{b}|}$

$$|\vec{a}| = \sqrt{1^2 + 2^2 + (-2)^2} = \sqrt{9} = 3$$

$$|\vec{b}| = \sqrt{4^2 + 0^2 + (-3)^2} = \sqrt{25} = 5$$

$$\vec{a}\cdot\vec{b} = (1)(4) + (2)(0) + (-2)(-3) = 4 + 0 + 6 = 10$$

$$\cos\theta = \frac{10}{(3)(5)} = \frac{10}{15} = \frac{2}{3}$$

$$\theta = \cos^{-1}\left(\frac{2}{3}\right) \approx 48.2^\circ$$