12.5 Lines and planes in space

Remark: We will (more or less) confuse points and vectors.

Given pts $P(3,7,-1)$ $Q(1,3,5)$

the vector $\vec{PQ}$ = displacement from P to Q

$$\vec{PQ} = \langle 1,3,5\rangle - \langle 3,7,-1\rangle$$
$$= \langle -2,-4,6\rangle.$$

Also given $P(3,7,-1)$ and the origin $(0,0,0)$

$$\vec{OP} = \langle 3,7,-1\rangle - \langle 0,0,0\rangle = \langle 3,7,-1\rangle$$

ex: [line in the plane]

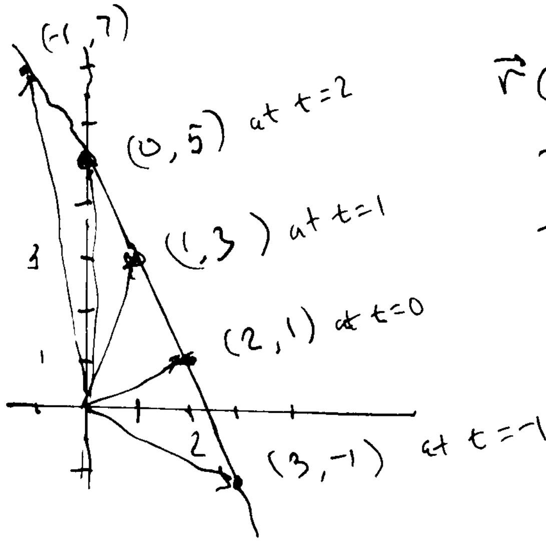
Draw these vectors: $\langle 2,1\rangle + (-1)\langle -1,2\rangle = \langle 3,-1\rangle$

$\langle 2,1\rangle$

$\langle 2,1\rangle + \langle -1,2\rangle = \langle 1,3\rangle$

$\langle 2,1\rangle + 2\langle -1,2\rangle = \langle 0,5\rangle$

$\langle 2,1\rangle + 3\langle -1,2\rangle = \langle -1,7\rangle$



$\vec{r}(t) = \langle 2,1\rangle + t\langle -1,2\rangle$

This "vector function" traces the line through $(2,1)$, parallel to the vector $\langle -1,2\rangle$

Vector form of a
~~Parametric~~ Line in Space

$$\boxed{\vec{r}(t) = \vec{r}_0 + t\,\vec{v}}$$

through $\vec{r}_0$ in the direction of $\vec{v}$

ex: a) Find a vector equation of a line through (3, 0, 5) in the direction $\vec{v} = \langle 0, -2, 1 \rangle$.

$$\vec{r}(t) = \langle 3, 0, 5 \rangle + t\langle 0, -2, 1 \rangle$$
$$= \langle 3, 0, 5 \rangle + \langle 0, -2t, t \rangle$$
$$= \langle 3, -2t, 5+t \rangle$$

b) What parametric equations describe this curve (line).

$$\begin{cases} x(t) = 3 \\ y(t) = \quad -2t \\ z(t) = 5 + t \end{cases}$$

ex: For the parametric line

$$\begin{cases} x = 3 + 7t \\ y = 1 - 4t \\ z = 6 + t \end{cases}$$

Find the "symmetric equations" by eliminating t.

[Note: The vector form of this line would be
$$\vec{r}(t) = \langle 3, 1, 6 \rangle + t\langle 7, -4, 1 \rangle$$]

ex (cont'd)

$$t = \boxed{\frac{x-3}{7} = \frac{y-1}{-4} = \frac{z-6}{1}}$$ symmetric equations

Remark (1) In practice the parametric equations are good for generating points on the line.

e.g: Find three points the line. Take $t=0,1,2$.

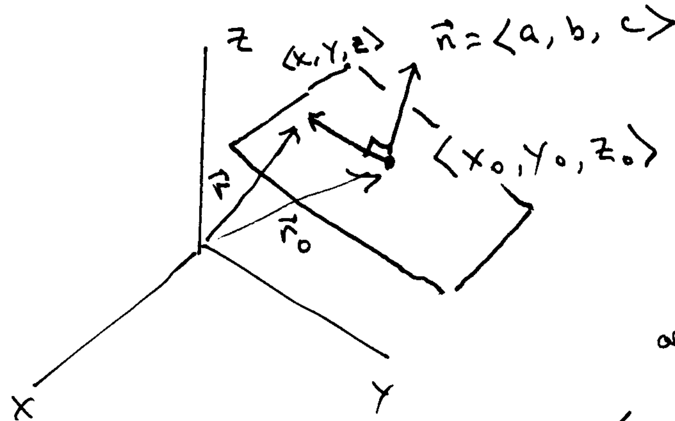
1st pt: $(3, 1, 6)$
2nd pt: $(10, -3, 7)$
3rd pt: $(17, -7, 8)$

(2) The symmetric equations are good for determining if a point lies on a given line.

e.g. Is $\underset{(x,y,z)}{} = (-4, -3, 5)$ on the line?

$$-1 = \frac{(-4)-3}{7} \overset{?}{=} \frac{(-3)-1}{-4} = +1$$ No.

Planes in space



$$\vec{r} \cdot \vec{n} = 0$$

or $\vec{n} \cdot \vec{r} = 0$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

Equation of Plane
through (x_0, y_0, z_0)
with <u>normal</u> <u>vector</u> $\langle a, b, c \rangle$.

$$\boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0}$$

ex: Find an eqn of the plane through $(1, 3, -5)$
normal to $\vec{n} = \langle 2, 3, 6 \rangle$.

$$2(x - 1) + 3(y - 3) + 6(z + 5) = 0$$

<u>Note</u>: $(x, y, z) = (1, 3, -5)$ satisfies this equation.

[The following was added after class ended]

<u>Remark</u>: This "point-normal vector" form of the equation of the plane is NOT unique to the plane, for any (nonzero) scalar multiple of $\vec{n}$ would describe the same plane. Likewise, any <u>other</u> point in the plane could be used in place of (x_0, y_0, z_0) and it would also give rise to an <u>equivalent</u> equation, so the same plane.