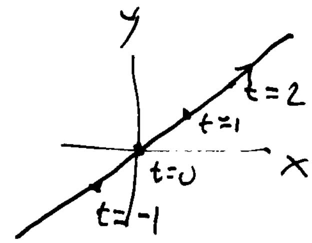
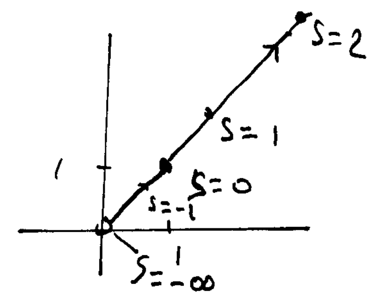
a little more on 13.1 : Different parametrizations of the same curve

ex: $\begin{cases} x = t \\ y = t \end{cases}$ or $\vec{r}(t) = \langle t, t \rangle$



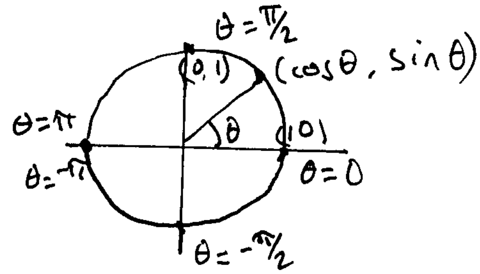
ex: $\begin{cases} x = e^s \\ y = e^s \end{cases}$ or $\vec{r}(s) = \langle e^s, e^s \rangle$



Two ways to parametrize a unit circle

ex : $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$

$-\pi \le \theta \le \pi$



Now behold some trig identities:

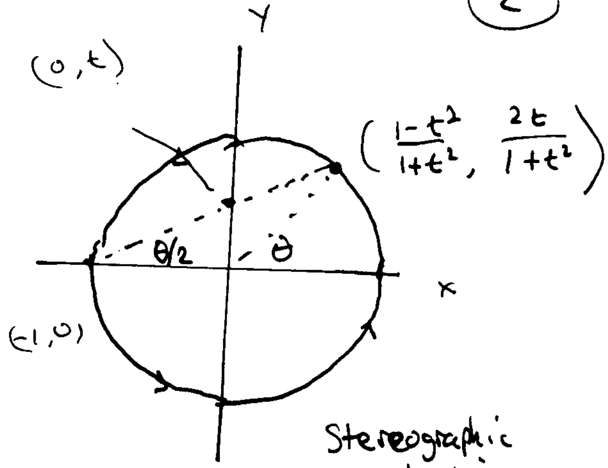
$$\cos\theta = \frac{\cos 2\left(\frac{\theta}{2}\right)}{1} = \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} \cdot \frac{\frac{1}{\cos^2\frac{\theta}{2}}}{\frac{1}{\cos^2\theta/2}} = \frac{\frac{\cos^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2}} - \frac{\sin^2\theta/2}{\cos^2\theta/2}}{\frac{\cos^2\frac{\theta}{2}}{\cos^2\theta/2} + \frac{\sin^2\theta/2}{\cos^2\theta/2}}$$

$$= \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} = \frac{1-t^2}{1+t^2} \quad \text{where } t = \tan\theta/2$$

Likewise, $\sin\theta = \dfrac{\sin 2\left(\frac{\theta}{2}\right)}{1} = \dfrac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \cdots = \dfrac{2\tan\theta/2}{1+\tan^2\theta/2} = \dfrac{2t}{1+t^2}$

ex: $\begin{cases} x = \dfrac{1-t^2}{1+t^2} \\ y = \dfrac{2t}{1+t^2} \end{cases}$

$-\infty < t < \infty$



Stereographic projection

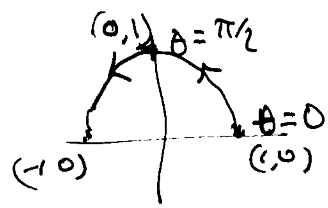
Remarks: about parametrizing

① If you replace t with $-t$, it makes the curve "runs backwards."

② If you replace t with $t+3$, it "advances" the tracing of the curve by 3 seconds.

③ If you replace t with $2t$, you trace the curve twice as fast.

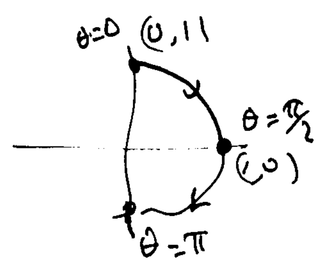
ex: $\begin{cases} x = \cos\theta \\ y = \sin\theta \end{cases}$



New parametrization gotten by (1st) advance by $\pi/2$ by replacing θ with $\theta + \pi/2$

(2nd) replace θ with $-\theta$

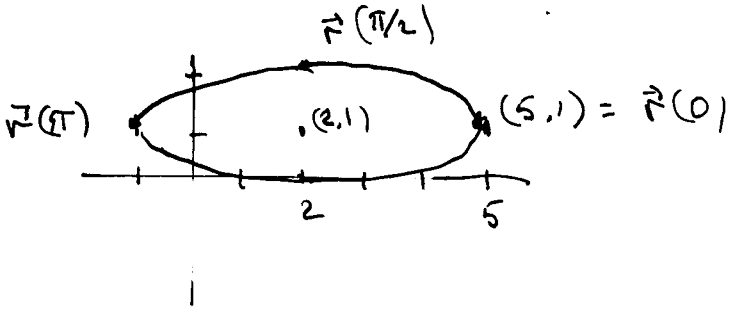
$\begin{cases} x = \cos(-\theta + \pi/2) = \sin\theta \\ y = \sin(-\theta + \pi/2) = \cos\theta \end{cases}$



Remark: Modifying x or y will change the shape of the curve:

ex: $\begin{cases} x = 3\cos\theta + 2 \\ y = \sin\theta + 1 \end{cases}$ or $\vec{r}(\theta) = \langle 3\cos\theta + 2, \sin\theta + 1 \rangle$

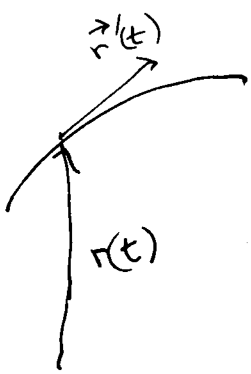
An ellipse with center $= (2,1)$, semimajor axis $= 3$, semiminor axis $= 1$.



13.2 Derivatives and integrals

Defn: $\vec{r}\,'(t) = \dfrac{d\vec{r}}{dt} = \lim\limits_{h \to 0} \dfrac{\vec{r}(t+h) - \vec{r}(t)}{h}$

Remark: $\vec{r}\,'(t)$ will be a vector tangent to the curve traced out by $\vec{r}(t)$.



Theorem: If $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$

then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t) \rangle$

ex: $\vec{r}(t) = \langle 3\cos t, 3\sin t, t \rangle$

$r'(t) = \langle -3\sin t, 3\cos t, 1 \rangle$

Theorem $\boxed{3}$: Sum rule, various product rules, chain rule.

[See the textbook]

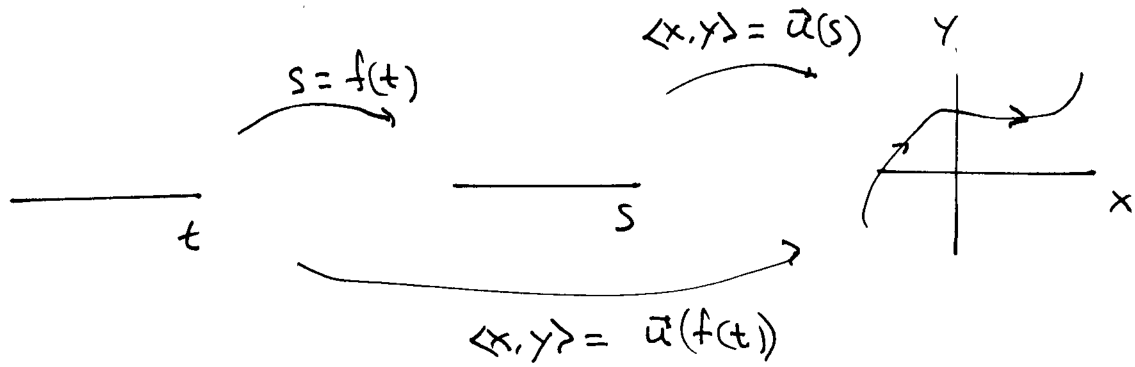
<u>ex</u> (of the <u>scalar</u> product rule) Find $\vec{r}'(t)$.

$$\vec{r}(t) = \langle e^{3t}\cos t,\ e^{3t}\sin t,\ e^{3t} \rangle$$
$$= e^{3t}\langle \cos t, \sin t, 1 \rangle$$

$$r'(t) = \frac{d}{dt}[e^{3t}]\langle \cos t, \sin t, 1\rangle + e^{3t}\frac{d}{dt}\langle \cos t, \sin t, 1\rangle$$
$$= 3e^{3t}\langle \cos t, \sin t, 1\rangle + e^{3t}\langle -\sin t, \cos t, 0\rangle$$
$$= e^{3t}\left[\langle 3\cos t, 3\sin t, 3\rangle + \langle -\sin t, \cos t, 0\rangle\right]$$
$$= e^{3t}\langle 3\cos t - \sin t,\ 3\sin t + \cos t,\ 3\rangle$$

ex (of our first generalization of the chain rule)

Theorem 3 #6: $\frac{d}{dt}\left[\vec{u}(f(t))\right] = f'(t)\,\vec{u}'(f(t))$



<u>ex</u>: $r(t) = \langle \cos t^2, \sin t^2 \rangle = \langle \cos s, \sin s\rangle$

where $s = t^2$

$$\frac{d\vec{r}}{dt} = \frac{d\vec{r}}{ds}\frac{ds}{dt} = \frac{ds}{dt}\frac{d\vec{r}}{ds} = 2t\langle -\sin s, \cos s\rangle$$
$$= 2t\langle -\sin t^2, \cos t^2\rangle$$

Defn: A curve is <u>smooth</u> if $\vec{r}\,'(t) \neq \vec{0}$.

Defn: For a smooth curve the unit tangent vector, $\vec{T}$, is defined by $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

ex: position $= \vec{r}(t) = \langle 3\cos t, 3\sin t, t\rangle$

velocity $= \vec{r}\,'(t) = \langle -3\sin t, 3\cos t, 1\rangle$

speed $= |\vec{r}\,'(t)| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 1^2}$

$= \sqrt{9(\sin^2 t + \cos^2 t) + 1} = \sqrt{9+1} = \sqrt{10}$

unit tangent $= \vec{T} = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \dfrac{1}{\sqrt{10}}\langle -3\sin t, 3\cos t, 1\rangle$

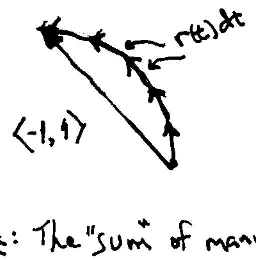
$= \left\langle -\frac{3}{\sqrt{10}}\sin t, \frac{3}{\sqrt{10}}\cos t, \frac{1}{\sqrt{10}}\right\rangle$

<u>Integrals</u>

$\int_a^b \vec{r}(t)\,dt$ can be calculated component-wise.

ex: $\vec{r}(t) = \langle -\sin t, \cos t\rangle$

$$\int_0^{\pi/2} \vec{r}(t)\,dt = \left\langle \int_0^{\pi/2} -\sin t\,dt, \int_0^{\pi/2} \cos t\,dt \right\rangle$$



$$= \langle \cos t, \sin t\rangle \Big|_0^{\pi/2}$$

$$= \langle 0, 1\rangle - \langle 1, 0\rangle = \langle -1, 1\rangle$$

<u>Remark</u>: The "sum" of many, gradually varying small vectors r(t)dt yields a resultant vector $\langle -1, 1\rangle$.