Supplemental notes on topics from 13.3: Curvature and 13.4: Velocity and Acceleration

Defn: For a vector function $\vec{r}(t)$, we define the arc length parameter s by

$$s(t) = \int_a^t |\vec{r}\,'(u)|\,du$$

Fact: By the Fundamental Theorem of Calculus,

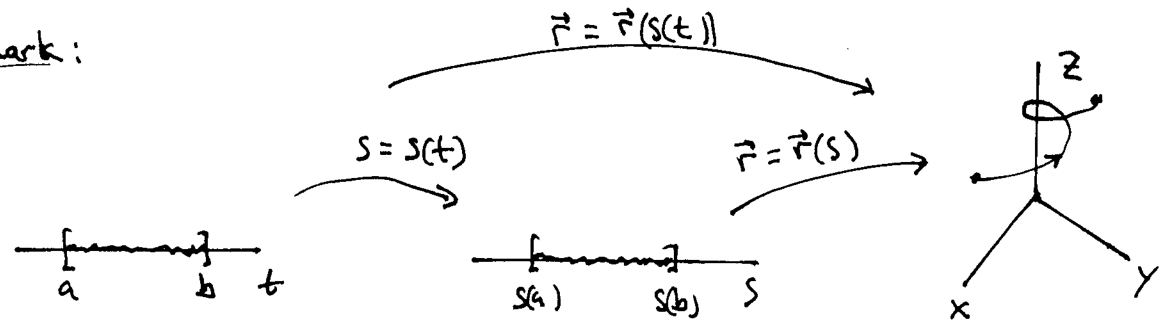
$$\frac{ds}{dt} = \frac{d}{dt}\left[\int_a^t |\vec{r}\,'(u)|\,du\right] = |\vec{r}\,'(t)|$$

Remark: Informally, if $\vec{r}(t)$ represents the position of a car on a highway as a function of t = time, then $\vec{r}\,'(t)$ = velocity (in meters/second), $|\vec{r}\,'(t)|$ = speed, $s(t)$ = the car's odometer reading (meters), so that

$$\frac{ds}{dt} = |\vec{r}\,'(t)| \quad \text{says, in effect}$$

$$\frac{d}{dt}[\text{odometer reading}] = \text{speedometer reading}.$$

Remark:



It's often hard to calculate s exactly, but often it is enough to know theoretically that $\vec{r}(t)$ can be written as a composition factoring through the arc length function $\vec{r} = \vec{r}(s(t))$ so that the Chain Rule applies.

<u>Defn</u>: The unit tangent vector, $\vec{T}$, is defined by

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

Solving for $\vec{r}\,'(t)$ and using that $\frac{ds}{dt} = |\vec{r}\,'(t)| = \text{speed}$ we have $\vec{r}\,'(t) = \frac{ds}{dt}\vec{T}$, <u>NOTE</u>: $|\vec{T}| = \frac{|\vec{r}\,'(t)|}{|\vec{r}\,'(t)|} = 1$.

<u>ex</u>: Take the standard parameterization of the circle of radius R, center at $(0,0)$:

$$\vec{r}(t) = \langle R\cos t,\ R\sin t\rangle \qquad \text{Then,}$$

$$\text{velocity} = \vec{r}\,'(t) = \langle -R\sin t,\ R\cos t\rangle$$

$$\text{Speed} = |\vec{r}\,'(t)| = \sqrt{(-R\sin t)^2 + (R\cos t)^2}$$
$$= \sqrt{R^2(\sin^2 t + \cos^2 t)} = R$$

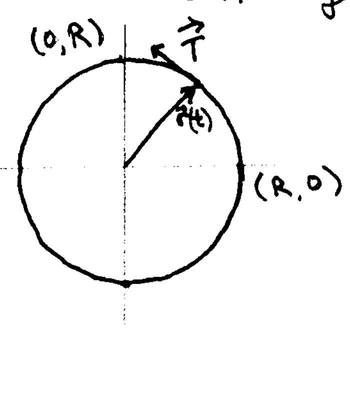
$$\text{arc length parameter} = s(t) = \int_0^t |\vec{r}\,'(u)|\,du = \int_0^t R\,du = Rt$$

<u>Remark</u>: Solving for t, this says $t = \frac{s}{R}$ so that

$$\vec{r}(s) = \left\langle R\cos\frac{s}{R},\ R\sin\frac{s}{R}\right\rangle$$

would be a way to parameterize the circle by the arclength parameter.

$$\text{unit tangent} = \vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{1}{R}\langle -R\sin t,\ R\cos t\rangle$$
$$= \langle -\sin t,\ \cos t\rangle$$



[end of example... for now]

Fact: $\vec{T}'$ is orthogonal to $\vec{T}$.

proof: $|\vec{T}|^2 = \vec{T}\cdot\vec{T} = 1$, Take derivatives of both sides and use the dot product Product Rule:

$$0 = \frac{d}{dt}[1] = \frac{d}{dt}[\vec{T}\cdot\vec{T}] = \frac{d\vec{T}}{dt}\cdot\vec{T} + \vec{T}\cdot\frac{d\vec{T}}{dt} = 2\left(\vec{T}\cdot\frac{d\vec{T}}{dt}\right)$$

Divide by 2: $0 = \vec{T}\cdot\frac{d\vec{T}}{dt}$ $\square$

Defn: Define the curvature, κ, of a curve defined by $\vec{r}(t)$ by

$$\kappa = \left|\frac{d\vec{T}}{ds}\right|$$

ex: [circle of radius R example continued] we calculated that

$$\vec{T}(t) = \langle -\sin t, \cos t\rangle$$

but since the arc length parameter is related to t by $t = \frac{s}{R}$ we have (note that R is a constant):

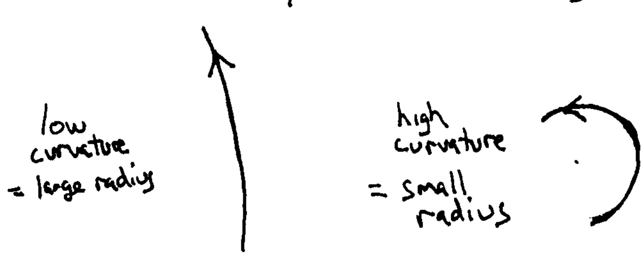
$$\vec{T}(s) = \left\langle -\sin\frac{s}{R}, \cos\frac{s}{R}\right\rangle \quad \text{so that}$$

$$\frac{d\vec{T}}{ds} = \left\langle -\frac{1}{R}\cos\frac{s}{R}, -\frac{1}{R}\sin\frac{s}{R}\right\rangle \quad \text{and}$$

$$\kappa = \left|\frac{dT}{ds}\right| = \sqrt{\left(\frac{1}{R}\right)^2\left(\cos^2\frac{s}{R} + \sin^2\frac{s}{R}\right)} = \frac{1}{R}$$

Remark: If you are driving on a circular road of radius R meters, your direction is changing at a rate of $\frac{1}{R}$ radians/meter of travel,

that is, curvature $= \frac{1}{\text{radius}}$ for a circle (or for a curve locally approximated by a circle).



<u>Remark</u>: The reason for using the arc length parameter s in the definition of curvature is so that κ does not depend on the particular <u>parametrization</u> of the curve; rather, only on the <u>geometry</u> of the curve.

<u>Defn</u>: If curvature, κ, is not zero, define the <u>principle</u> <u>unit</u> normal <u>vector</u> $\vec{N}$ of the curve by

$$\vec{N}(s) = \frac{\frac{d\vec{T}}{ds}}{\left|\frac{d\vec{T}}{ds}\right|} = \frac{\frac{d\vec{T}}{ds}}{\kappa}$$

<u>Properties</u>: 1) $|\vec{N}| = 1$ because $|\vec{N}| = \left|\frac{dT/ds}{|dT/ds|}\right| = \frac{|dT/ds|}{|dT/ds|} = 1$

2) $\vec{T} \cdot \vec{N} = 0$ because

$$\vec{T} \cdot \vec{N} = \vec{T} \cdot \frac{\frac{d\vec{T}}{ds}}{\left|\frac{dT}{ds}\right|} = \frac{1}{\kappa}\left(T \cdot \frac{d\vec{T}}{ds}\right) = \frac{1}{\kappa}(0) = 0$$

3) $\frac{d\vec{T}}{ds} = \kappa \vec{N}$ because this results from solving the equation $\vec{N} = \frac{d\vec{T}/ds}{\kappa}$ for $\frac{d\vec{T}}{ds}$

4) If t is a parameter which is <u>not</u> necessarily the arc length parameter, then by the Chain Rule,

$$\vec{T}'(t) = \frac{d\vec{T}}{dt} = \frac{ds}{dt}\frac{d\vec{T}}{ds} = \frac{ds}{dt}\kappa\vec{N}$$

Now, if we take the length of both sides of the equation in Property 4) above, we get

$$|\vec{T}'(t)| = \left|\frac{d\vec{T}}{dt}\right| = \left|\frac{ds}{dt}\,\kappa\,\vec{N}\right| = \left(\frac{ds}{dt}\right)\kappa\,\cancel{|\vec{N}|}^{\,1} = \left(\frac{ds}{dt}\right)\kappa$$

Solving for κ we get

$$\boxed{\kappa = \frac{|d\vec{T}/dt|}{ds/dt} = \frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}}$$

"First (useful) curvature formula"

<u>Remark</u>: I call this the "first useful curvature formula" because it does <u>not</u> require us to calculate the arc length parameter; a good thing.
This is the formula to use if you have already calculated the unit tangent vector $\vec{T}(t)$ at some point.

Find $\vec{T}$ and $\vec{N}$ and κ and all that.

ex: For $\vec{r}(t) = \langle 4\cos^3 t, \ 4\sin^3 t \rangle \qquad 0 \le t \le \frac{\pi}{2}$

$$= 3\langle \cos t, \sin t \rangle + \langle \cos 3t, -\sin 3t \rangle$$

velocity $= \vec{r}\,'(t) = \langle -12\cos^2 t \sin t, \ 12 \sin^2 t \cos t \rangle$

$= 12 \cos t \sin t \langle -\cos t, \sin t \rangle = 6 \sin 2t \langle -\cos t, \sin t \rangle$

speed $= \frac{ds}{dt} = |\vec{r}\,'(t)| =$

$$= \left| 12 \cos t \sin t \langle -\cos t, \sin t \rangle \right|$$

$$= 12 \cos t \sin t \sqrt{(-\cos t)^2 + (\sin t)^2} = 12 \cos t \sin t$$

using: $|c\vec{v}| = |c|\,|\vec{v}|$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|r'(t)|} = \frac{1}{12\cos t \sin t} 12 \cos t \sin t \langle -\cos t, \sin t \rangle$$

$$= \langle -\cos t, \sin t \rangle$$

$$\vec{T}\,'(t) = \frac{d\vec{T}}{dt} = \langle \sin t, \cos t \rangle$$

$$|\vec{T}\,'(t)| = \sqrt{\sin^2 t + \cos^2 t} = 1$$

$$\kappa = \frac{|\vec{T}\,'|}{|\vec{r}\,'|} = \frac{1}{12 \cos t \sin t} = \frac{1}{6(2 \sin t \cos t)} = \frac{1}{6 \sin 2t}$$

so Radius of the osculating circle $= 6 \sin 2t$

$$\vec{N} = \frac{\vec{T}\,'}{|T'|} = \frac{\langle \sin t, \cos t \rangle}{1} = \langle \sin t, \cos t \rangle$$

What about acceleration? $\vec{r}\,''(t) = \frac{d}{dt}\left[6 \sin 2t \langle -\cos t, \sin t \rangle \right]$

$$= \cancel{12 \cos 2}\ \frac{d}{dt}[6 \sin 2t] \langle -\cos t, \sin t \rangle + 6 \sin 2t \frac{d}{dt} \langle -\cos t, \sin t \rangle$$

$$= \underbrace{12 \cos 2t}_{} \underbrace{\langle -\cos t, \sin t \rangle}_{\vec{T}} + \underbrace{6 \sin 2t}_{} \underbrace{\langle \sin t, \cos t \rangle}_{\vec{N}}$$

$$\frac{d}{dt}[\text{speed}] = \underbrace{\frac{d^2 s}{dt^2}}_{a_T} \qquad \underbrace{\left(\frac{ds}{dt}\right)^2 \kappa}_{a_N}$$

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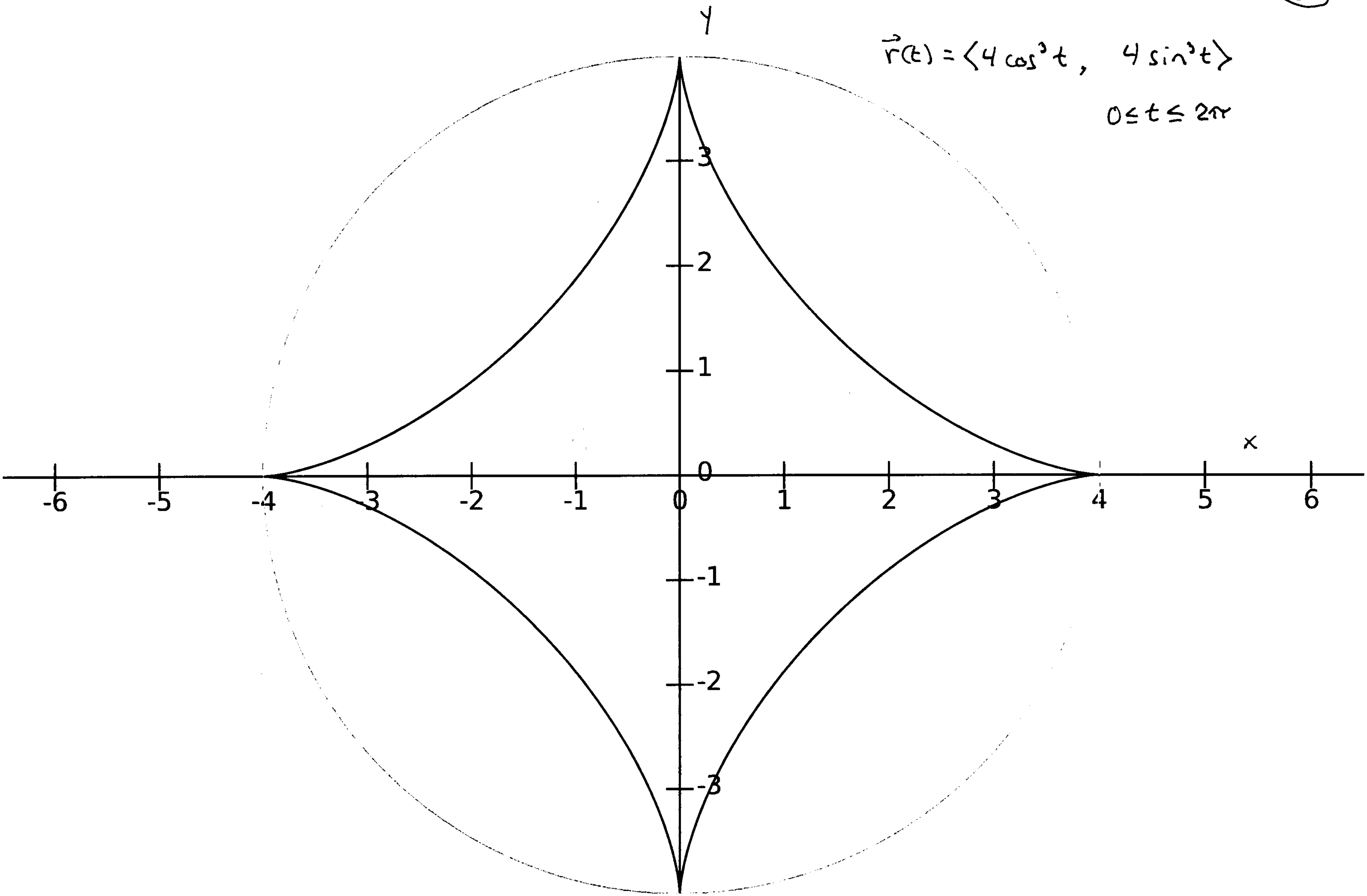
$\vec{r}(t) = \langle 4\cos^3 t, \ 4\sin^3 t \rangle$

$0 \le t \le 2\pi$



<u>Theorem</u> [the Pinacle of Chapter 13]

$$\text{acceleration} = \boxed{\vec{r}''(t) = \underbrace{\frac{d^2s}{dt^2}\vec{T}}_{\text{controlled by your feet}} + \underbrace{\left(\frac{ds}{dt}\right)^2 \kappa \vec{N}}_{\text{controlled by your hands}}}$$

as a driver: controlled by your feet; controlled by your hands

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Added after class: <u>Proof</u> of theorem:

Because $\vec{T} = \dfrac{\vec{r}'(t)}{|\vec{r}'(t)|}$ we have $\vec{r}'(t) = |\vec{r}'(t)|\,\vec{T}$

or, equivalently, $\dfrac{d\vec{r}}{dt} = \dfrac{ds}{dt}\vec{T}$.

Take derivatives of both sides of this equation, using the (scalar) product rule,

$$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left[\frac{ds}{dt}\vec{T}\right] = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\frac{d\vec{T}}{dt}$$

From Property 4) on page (4) of today's notes, $\dfrac{d\vec{T}}{dt} = \dfrac{ds}{dt}\kappa\vec{N}$.

Substituting into the equation above, we get

$$\frac{d^2\vec{r}}{dt^2} = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N} \qquad \square$$

<u>Note</u> that $\dfrac{ds}{dt}$ = speed = · while $\dfrac{d^2s}{dt^2} = \dfrac{d}{dt}[\text{speed}]$ = second derivative of the arc length parameter.

Informally, $\dfrac{d^2s}{dt^2}$ is a measure of how hard you're stepping on the accelerator (when $\dfrac{d^2s}{dt^2} > 0$) or on the brake (when $\dfrac{d^2s}{dt^2} < 0$)

<u>Defn</u>: $a_T = \dfrac{d^2s}{dt^2}$ = the <u>tangential</u> <u>component</u> <u>of</u> <u>acceleration</u>

$a_N = \left(\dfrac{ds}{dt}\right)^2 \kappa$ = the <u>normal</u> <u>component</u> <u>of</u> <u>acceleration</u> .

Theorem:

$$\boxed{\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}}$$

"Second (useful) curvature formula"

proof: Because

$$\vec{r}\,'(t) = \frac{ds}{dt}\vec{T}$$

and

$$\vec{r}\,''(t) = \frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N},$$

$$\vec{r}\,' \times \vec{r}\,'' = \frac{ds}{dt}\vec{T} \times \left[\frac{d^2s}{dt^2}\vec{T} + \left(\frac{ds}{dt}\right)^2 \kappa \vec{N}\right]$$

$$= \left(\frac{ds}{dt}\right)\left(\frac{d^2s}{dt^2}\right)\cancelto{\vec{0}}{(\vec{T} \times \vec{T})} + \left(\frac{ds}{dt}\right)^3 \kappa (\vec{T} \times \vec{N})$$

$$= \left(\frac{ds}{dt}\right)^3 \kappa (\vec{T} \times \vec{N}), \quad \text{for } |\vec{T} \times \vec{T}| = |\vec{T}||\vec{T}| \sin 0 = 0.$$

$$|\vec{r}\,' \times \vec{r}\,''| = \left(\frac{ds}{dt}\right)^3 \kappa \, |\vec{T} \times \vec{N}| = \left(\frac{ds}{dt}\right)^3 \kappa,$$

$$\text{for } |\vec{T} \times \vec{N}| = |\vec{T}||\vec{N}| \sin \tfrac{\pi}{2} = 1.$$

Solving for κ, and using that $\frac{ds}{dt} = |\vec{r}\,'|$, we get

$$\kappa = \frac{|\vec{r}\,' \times r''|}{\left(\frac{ds}{dt}\right)^3} = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \qquad \square$$

Remark: If $\vec{r}$ is measured in meters and t is measured in seconds, the units behave as expected

- units for $\dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} = \dfrac{\frac{m}{s} \cdot \frac{m}{s^2}}{\frac{m^3}{s^3}} = \dfrac{1}{m}$ = units for curvature.