14.1 Functions of two or three variables

ex: $f(x,y) = \sqrt{49 - x^2 - y^2}$

$$f(2,-3) = \sqrt{49 - (2)^2 - (-3)^2}$$
$$= \sqrt{49 - 4 - 9} = \sqrt{36} = 6$$

$$f(0,0) = \sqrt{49 - 0^2 - 0^2} = 7$$

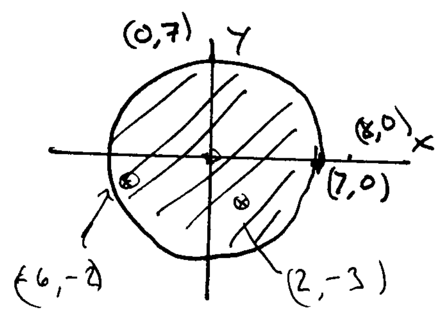
$$f(-6,-2) = \sqrt{49 - (-6)^2 - (-2)^2} = \sqrt{49 - 36 - 4}$$
$$= \sqrt{9} = 3$$

But $f(8,0) = \sqrt{49 - 8^2 - 0^2} = \sqrt{-15}$ is not real,

so $(8,0)$ is not in the domain of f.

What are the "legal" inputs? Solve:

$$49 - x^2 - y^2 \geq 0$$
or $49 \geq x^2 + y^2$ or $x^2 + y^2 \leq 49$



domain $= \{(x,y) \mid x^2 + y^2 \leq 49\}$

= disk of radius ~~49~~ 7 centered at origin

What about the range (and the graph)?

Graph:

$$z = \sqrt{\cancel{4}49 - x^2 - y^2}$$

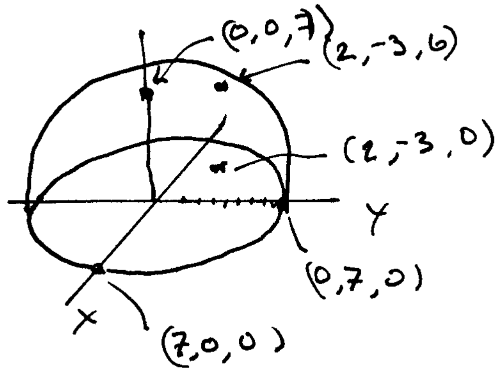
$\left.\begin{matrix} x \\ y \end{matrix}\right\}$ = independent variable

z = dependent variable

Square both sides:

$$z^2 = 49 - x^2 - y^2$$

$$x^2 + y^2 + z^2 = 49$$ ← sphere of radius 7, center at (0,0,0)

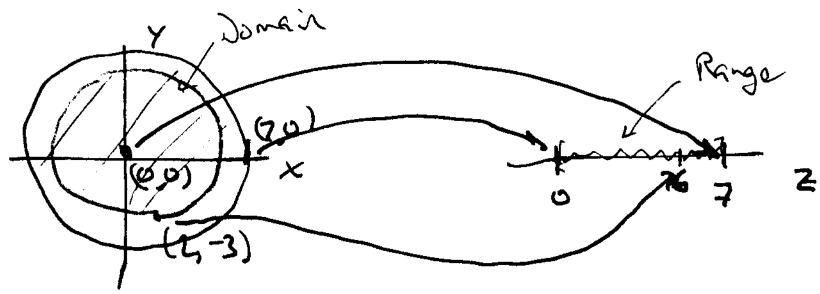


Range $= \{z \mid 0 \le z \le 7\}$

$= [0, 7]$

= set of all possible values of the dependent variable.

mapping:



For this function, what is the level curve for $z=6$?

Set $z=6$ and "solve" for x and y:

$$6 = \sqrt{49 - x^2 - y^2}$$

$$36 = 49 - x^2 - y^2$$

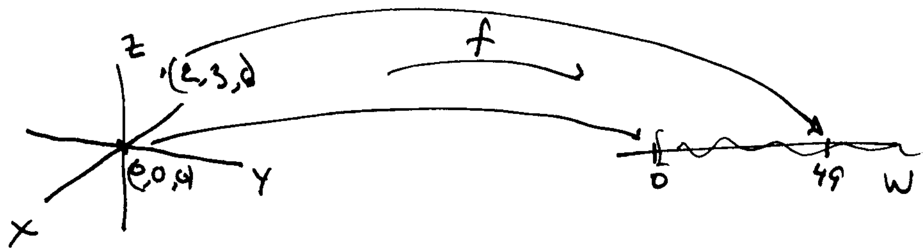
$$x^2 + y^2 = 49 - 36$$

$$x^2 + y^2 = 13$$ ← a circle of radius $\sqrt{13}$, center (0,0)

≈ 3.6

ex [function of three variables]

$$w = f(x,y,z) = x^2 + y^2 + z^2$$



Range $= [0,\infty)$

$$f(0,0,0) = 0$$

$$f(2,3,6) = 2^2 + 3^2 + 6^2 = 4 + 9 + 36 = 49$$

$$f(10^6, 10^6, 10^6) = 3 \cdot 10^{12}$$

what ordered triples map to 49? Set $w = 49$:

$$x^2 + y^2 + z^2 = 49$$ ← sphere of radius 7. which includes $(2,3,6)$

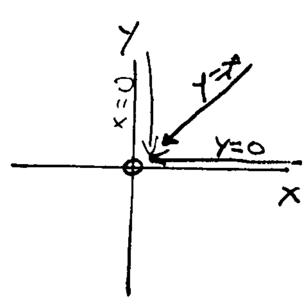
Remark: The level surfaces for this function will be spheres centered at $(0,0,0)$.

14.2 Limits and continuity (we're cutting corners here)

ex: [How limits can not exist]

$$z = f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

Does $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ exist?



i) approach $(0,0)$ along the x-axis (so $y=0$)

$$\lim_{(x,0)\to(0,0)} \frac{x^2 - 0^2}{x^2 + 0^2} = \lim_{x\to 0} 1 = 1$$

ii) Approach $(0,0)$ along the y-axis (so $x=0$).

$$\lim_{(0,y)\to(0,0)} \frac{0^2-y^2}{0^2+y^2} = \lim_{y\to 0} -1 = -1$$

Since $1 \neq -1$, the limit doesn't exist.

ex: $f(x,y) = \dfrac{xy}{x^2+y^2}$ Likewise $\lim_{(x,y)\to(0,0)} f(x,y)$ will not exist.

i) $$\lim_{(x,0)\to(0,0)} \frac{x\cdot 0}{x^2+0^2} = \lim_{x\to 0} 0 = 0$$

ii) $$\lim_{(0,y)\to(0,0)} \frac{0\cdot y}{0^2+y^2} = \lim_{y\to 0} 0 = 0$$

iii) Approach along the line $x=y$.

$$\lim_{(x,x)\to(0,0)} \frac{x\cdot x}{x^2+x^2} = \lim_{x\to 0} \frac{x^2}{2x^2} = \lim_{x\to 0} \frac{1}{2} = \frac{1}{2}$$

So, since $\frac{1}{2} \neq 0$ the limit does not exist.

14.3 Partial Derivatives (a quick introduction)

<u>Defn</u>: If f is a function of two variables, the <u>partial derivatives</u> of f are

$$f_x(x,y) = \frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$f_y(x,y) = \frac{\partial f}{\partial y} = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

Remark: By staring at the definition, we see that we, in fact, already know how to calculate partial derivatives.

ex: $f(x,y) = x^3 + 5x^2y + 7y^2$

$f_x(x,y) = 3x^2 + 10xy$

$f_y(x,y) = 5x^2 + 14y$

ex: $f(x,y) = ye^{3xy}$

$f_x(x,y) = \frac{\partial}{\partial x}\left[ye^{3xy}\right] = y\,\frac{\partial}{\partial x}\left[e^{3xy}\right]$ ← chain rule

$= y\,e^{3xy}\cdot\underbrace{\frac{\partial}{\partial x}[3xy]}_{3y} = \boxed{3y^2e^{3xy}}$

$f_y(x,y) = \frac{\partial}{\partial y}\left[y\,e^{3xy}\right]$ ← product rule

$= \frac{\partial y}{\partial y}\cdot e^{3xy} + y\,\frac{\partial}{\partial y}\left[e^{3xy}\right]$ ← chain rule

$= 1\cdot e^{3xy} + y\cdot e^{3xy}\cdot\frac{\partial}{\partial y}[3xy]$

$= e^{3xy} + ye^{3xy}\cdot 3x\,\cancel{\frac{\partial y}{\partial y}}\to 1$

$= \boxed{e^{3xy} + 3xye^{3xy}}$

or $(1+3xy)e^{3xy}$