14.4 Tangent planes and approximation: "The Total Differential"

Idea: Functions $z=f(x,y)$ which are differentiable have graphs which are approximated by their tangent plane.

Recall: $z=f(x,y)$ is differentiable at (x_0,y_0) if

$$\underbrace{\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y}_{\text{equation of the tangent plane}} + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$

Equation of Tangent plane: $$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

Recall: Eqn of a plane through (x_0,y_0,z_0) with normal vector $\langle f_x(x_0,y_0), f_y(x_0,y_0), -1\rangle$ would be

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$

14.4 modified 21) Find the linear approximation of the function
$$f(x,y) = \sqrt{x^2+y^2} \quad \text{at} \quad (x_0,y_0) = (3,2)$$
and use this to approximate $f(3.02, 1.97)$.

$$\Delta z = f(3.02, 1.97) - f(3,2)$$

$$\approx f_x(3,2)\,\Delta x + f_y(3,2)\,\Delta y$$

where $\Delta x = 3.02 - 3 = 0.02$

and $\Delta y = 1.97 - 2 = -0.03$

$$f_x(x,y) = \frac{\partial}{\partial x}\left[(x^2+y^2)^{1/2}\right] = \frac{1}{2}(x^2+y^2)^{-1/2}\cdot\underbrace{\frac{\partial}{\partial x}[x^2+y^2]}_{2x}$$

$$= x(x^2+y^2)^{-1/2} = \frac{x}{\sqrt{x^2+y^2}}$$

Similarly, $f_y(x,y) = \dfrac{y}{\sqrt{x^2+y^2}}$

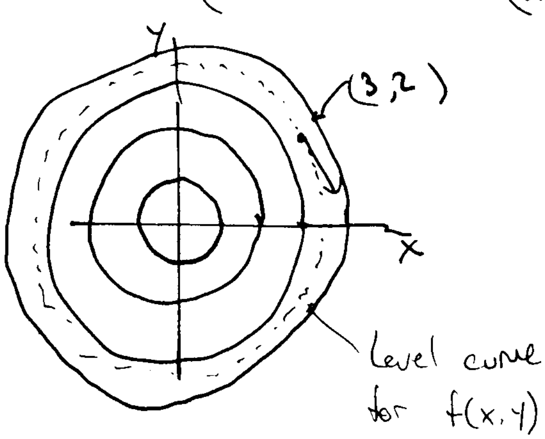
At $(3,2)$; $f_x(3,2) = \dfrac{3}{\sqrt{3^2+2^2}} = \dfrac{3}{\sqrt{13}}$

and $f_y(3,2) = \dfrac{2}{\sqrt{13}}$

$$\Delta z = f_x(3,2)\cdot\Delta x + f_y(3,2)\cdot\Delta y$$

$$= \left(\frac{3}{\sqrt{13}}\right)(0.02) + \left(\frac{2}{\sqrt{13}}\right)(-0.03) = \frac{(3)(.02)-(2)(.03)}{\sqrt{13}} = 0$$

Visual explanation:



$(\Delta x, \Delta y) = \langle 0.02, -0.03\rangle$

This happens to be tangent to the level curve.

<u>Defn</u>: If $z = f(x, y)$, the <u>differential</u> of f, (or the <u>total differential</u>), denoted dz or df, is

$$dz = df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy$$

$$= f_x(x,y)\, dx + f_y(x,y)\, dy$$

<u>Notation</u>: $dx = \Delta x$, $dy = \Delta y$.

<u>ex</u>: (#12) $z = f(x,y) = x^3 y^4$

$$dz = \frac{\partial}{\partial x}\left[x^3 y^4\right] \cdot dx + \frac{\partial}{\partial y}\left[x^3 y^4\right] \cdot dy$$

$$\underbrace{dz}_{\text{small change in elevation}} = \underbrace{3x^2 y^4}_{\text{slope in the east-west direction}}\ \underbrace{dx}_{\text{How much you're being nudged in the east-west direction}} + \underbrace{4x^3 y^3}_{\text{slope in north-south direction}}\ \underbrace{dy}_{\text{How far you're nudged in the north-south direction}}$$

14.4 26) Find du if $u = (x^2 + 3y^2)^{1/2}$

$$du = \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

$$= \frac{1}{2}(x^2+3y^2)^{-1/2} \cdot \frac{\partial}{\partial x}[x^2+3y^2]\,dx$$

$$+ \frac{1}{2}(x^2+3y^2)^{-1/2} \cdot \frac{\partial}{\partial y}[x^2+3y^2]\,dy$$

$$= \frac{1}{\cancel{2}}(x^2+3y^2)^{-1/2}\,\cancel{2}x\,dx + \frac{1}{\cancel{2}}(x^2+3y^2)^{-1/2} \cdot \overset{3}{\cancel{6}}y\,dy$$

$$= \frac{x\,dx + 3y\,dy}{\sqrt{x^2+3y^2}}$$

34) 

$V = \pi r^2 h$ Estimate the metal in the can

if $h = 10$ cm, $2r = 4$ cm

and $dh = 0.1$ cm, $dr = 0.05$ cm

answer: $dV = \frac{\partial V}{\partial r}\,dr + \frac{\partial V}{\partial h}\,dh$ ...

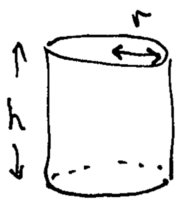
[see next page]

Added after the end of class:

Another attempt at

14.4 #34) Use differentials to estimate the amount of metal in a closed cylindrical can which is 10 cm high and 4 cm in diameter if the metal in the top and bottom are [each] 0.1 cm thick and the metal in the sides is 0.05 cm thick.

The volume of metal $=$ outer volume of can $-$ inner volume of can $= \Delta V$



where $V = \pi r^2 h$, for $r =$ radius, $h =$ height.

Using that $\Delta V \approx dV$ and

$$dV = \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = 2\pi r h\, dr + \pi r^2\, dh$$

where we want to take

$h = 10$ cm, $r = 2$ cm

$dh =$ thickness of top $+$ thickness of bottom $= 0.1 + 0.1 = 0.2$ cm

$dr =$ thickness of side $= 0.05$ cm

Then $dV = 2\pi\,(2\text{ cm})(10\text{ cm})(0.05\text{ cm}) + \pi\,(2\text{ cm})^2(0.2\text{ cm})$

$= 2\pi\text{ cm}^3 + 0.8\pi\text{ cm}^3$

$= 2.8\pi\text{ cm}^3 \approx \boxed{8.8\text{ cm}^3}$