14.6 Directional Derivatives and the Gradient Vector

Defn: A direction is represented by a unit vector $\vec{u} = \langle a, b \rangle$. That is $|\vec{u}| = \sqrt{a^2+b^2} = 1$.

Defn: If f is a differentiable function at (x_0, y_0) then the directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0+ha,\, y_0+hb) - f(x_0, y_0)}{h}.$$

Thm 3: $$D_{\vec{u}} f(x_0, y_0) = f_x(x_0,y_0)\, a + f_y(x_0,y_0)\, b$$
$$= \langle f_x(x_0,y_0),\, f_y(x_0,y_0) \rangle \cdot \langle a, b \rangle$$
$$= \nabla f(x_0, y_0) \cdot \vec{u}$$

where [Defn:] $\nabla f(x_0, y_0)$ = gradient of f at (x_0, y_0)
$= \langle f_x(x_0,y_0),\, f_y(x_0,y_0) \rangle$ ← a "vector field" [See Ch. 16]

14.6 8) $f(x,y) = \frac{y^2}{x} = x^{-1}y^2$

a) Find the gradient of f. $\nabla f(x,y) = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle$
$= \langle -\frac{y^2}{x^2}, \frac{2y}{x} \rangle$

8b) Evaluate the gradient at $(x,y) = (1,2)$

$$\nabla f\Big|_{(1,2)} = \nabla f(1,2) = \left\langle -\frac{2^2}{1^2}, \frac{2(2)}{1} \right\rangle = \langle -4, 4 \rangle$$

8c) Find the (instantaneous) rate of change of f at (1,2) in the direction of the vector $\vec{u} = \frac{1}{3}\langle 2, \sqrt{5} \rangle$

Check: Is $\vec{u}$ a <u>unit</u> vector?

$$|\vec{u}| = \left| \frac{1}{3}\langle 2, \sqrt{5} \rangle \right| = \frac{1}{3} \left| \langle 2, \sqrt{5} \rangle \right|$$

$$= \frac{1}{3}\sqrt{2^2 + \sqrt{5}^2} = \frac{1}{3}\sqrt{4+5} = \frac{1}{3}\sqrt{9} = 1. \text{ Yes!}$$

$$D_{\vec{u}} f(1,2) = \nabla f(1,2) \cdot \vec{u}$$

$$= \langle -4, 4 \rangle \cdot \frac{1}{3}\langle 2, \sqrt{5} \rangle$$

$$= \frac{1}{3}\left[(-4)(2) + (4)(\sqrt{5})\right] = \frac{1}{3}(-8 + 4\sqrt{5})$$

$$= -\frac{8}{3} + \frac{4}{3}\sqrt{5} = 0.315$$

12) $f(x,y) = \dfrac{x}{x^2+y^2}$ Find $D_{\vec{u}} f(1,2)$ where

$\vec{u}$ = unit vector in the direction of $\vec{v} = \langle 3, 5 \rangle$

<u>Method</u>: Same as previous problem <u>BUT</u> $\vec{u} = \dfrac{\vec{v}}{|\vec{v}|}$

That is $|\vec{v}| = \sqrt{3^2 + 5^2} = \sqrt{34}$ so $\vec{u} = \frac{1}{\sqrt{34}}\langle 3, 5 \rangle$.

<u>Remark</u>: $D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}$

$= |\nabla f(x,y)| \cancel{|\vec{u}|}^{\,1} \cos\theta$

$= |\nabla f(x,y)| \cos\theta$, where θ is the angle between $\vec{u}$ and the gradient of f.

So i) The rate of change is <u>maximized</u> when $\theta = 0$ that is, in the direction of ∇f.

ii) " <u>minimized</u> (~~equal~~ to $-\nabla f$) when $\theta = \pi = 180°$ that is, when $\vec{u}$ is in the opposite direction of gradient of f.

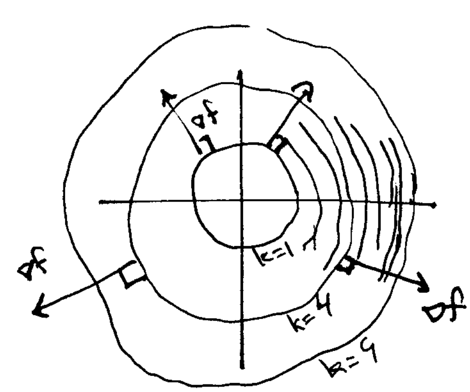
iii) $D_{\vec{u}} f(x,y) = 0$ when $\vec{u}$ is <u>orthogonal</u> to ∇f, so $\theta = 90° = \pi/2$. Then

$$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u} = 0 .$$

<u>Graphical</u> interpretation of the <u>gradient</u> of f:

∇f is a normal vector to each level curve of f.

<u>ex</u>: $f(x,y) = x^2 + y^2$ so $\nabla f(x,y) = \langle 2x, 2y \rangle$
$= 2\langle x, y \rangle$



∇f is always <u>normal</u> to the level curves, always pointing "uphill", and is <u>longer</u> where the level curves are close together (indicating faster increase of the function values)

Functions of three variable

Remark: $\nabla f(x,y,z) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

will be a vector normal to level surfaces of $f(x,y,z)$.

42) Consider $y = x^2 - z^2$. This can be written as

$x^2 - y - z^2 = 0$ so ~~it~~ this surface can be thought of as the level surface for $k=0$ of $f(x,y,z) = x^2 - y - z^2$.

a) Find an equation of the plane tangent to this surface at $(x,y,z) = (4,7,3)$.

$\nabla f(x,y,z) = \langle f_x, f_y, f_z \rangle = \langle 2x, -1, -2z \rangle$

At $(4,7,3)$, $\nabla f\Big|_{(4,7,3)} = \langle 2(4), -1, -2(3) \rangle$

$= \langle 8, -1, -6 \rangle$ ← normal vector to the surface, hence, a normal vector to the plane.

Tangent Plane: $8(x-4) - (y-7) - 6(z-3) = 0$

b) Find an equation of the normal line to the surface at $(4,7,3)$.

$\vec{r}(t) = \langle 4,7,3 \rangle + t\langle 8,-1,-6 \rangle$

or, parametric equations

$x = 4 + 8t$

$y = 7 - t$

$z = 3 - 6t$

14.7 Max/min for functions of two variables

Defn: $f(x,y)$ has a ~~local~~ local maximum at (a,b) if $f(x,y) \le f(a,b)$ when (x,y) is near (a,b).

Defn: A point (a,b) is a critical point if

either i) $f_x(a,b) = f_y(a,b) = 0$ (horizontal tangent plane)

or ii) $f_x(a,b)$ doesn't exist or $f_y(a,b)$ doesn't exist (no well-defined tangent plane)

Theorem. If there is a local max, it occurs at a critical point.

(But you can have a critical where there is no local max or min.)

Second derivative test: Suppose (a,b) is a critical point.

Let $$D = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix} = f_{xx}(a,b)\, f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$

= "product" of the concavities

At (a,b):

	$f_{xx} > 0$	$f_{xx} < 0$ [or use $f_{yy}(a,b)$]
$D > 0$	local min at (a,b)	local max at (a,b)
$D < 0$	saddle point at (a,b)	
$D = 0$	anything can happen	

Find and classify local max/min, saddle point

(14.7 9) $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

critical pts?

completed after the end of class. ↓

$0 = f_x(x,y) = 6xy - 12x = 6x(y-2) \Rightarrow x = 0$ or $y = 2$

$0 = f_y(x,y) = 3y^2 + 3x^2 - 12y$
$= 3(x^2 + y^2 - 4y)$

<u>Case 1</u>: If $x = 0$

then $0 = 3(0^2 + y^2 - 4y)$
$0 = y^2 - 4y = y(y-4)$
$\Rightarrow y = 0$ or $y = 4$

So two critical points are
$(x,y) = (0,0)$ and $(x,y) = (0,4)$

<u>Case 2</u>: If $y = 2$

then $0 = 3(x^2 + 2^2 - 4\cdot 2)$
So $0 = x^2 - 4 = (x-2)(x+2)$
$\Rightarrow x = 2$ or $x = -2$

Two more C.P.s are
$(x,y) = (2,2)$ and $(x,y) = (-2,2)$

Now, $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6y-12 & 6x \\ 6x & 6y-12 \end{vmatrix}$

We apply the 2nd derivative test at each critical point:

At $(0,0)$ $D = \begin{vmatrix} -12 & 0 \\ 0 & -12 \end{vmatrix} = 144 > 0$ and $f_{xx}(0,0) = -12 < 0$ so $\boxed{f(0,0) = 2 \text{ is a local maximum}}$

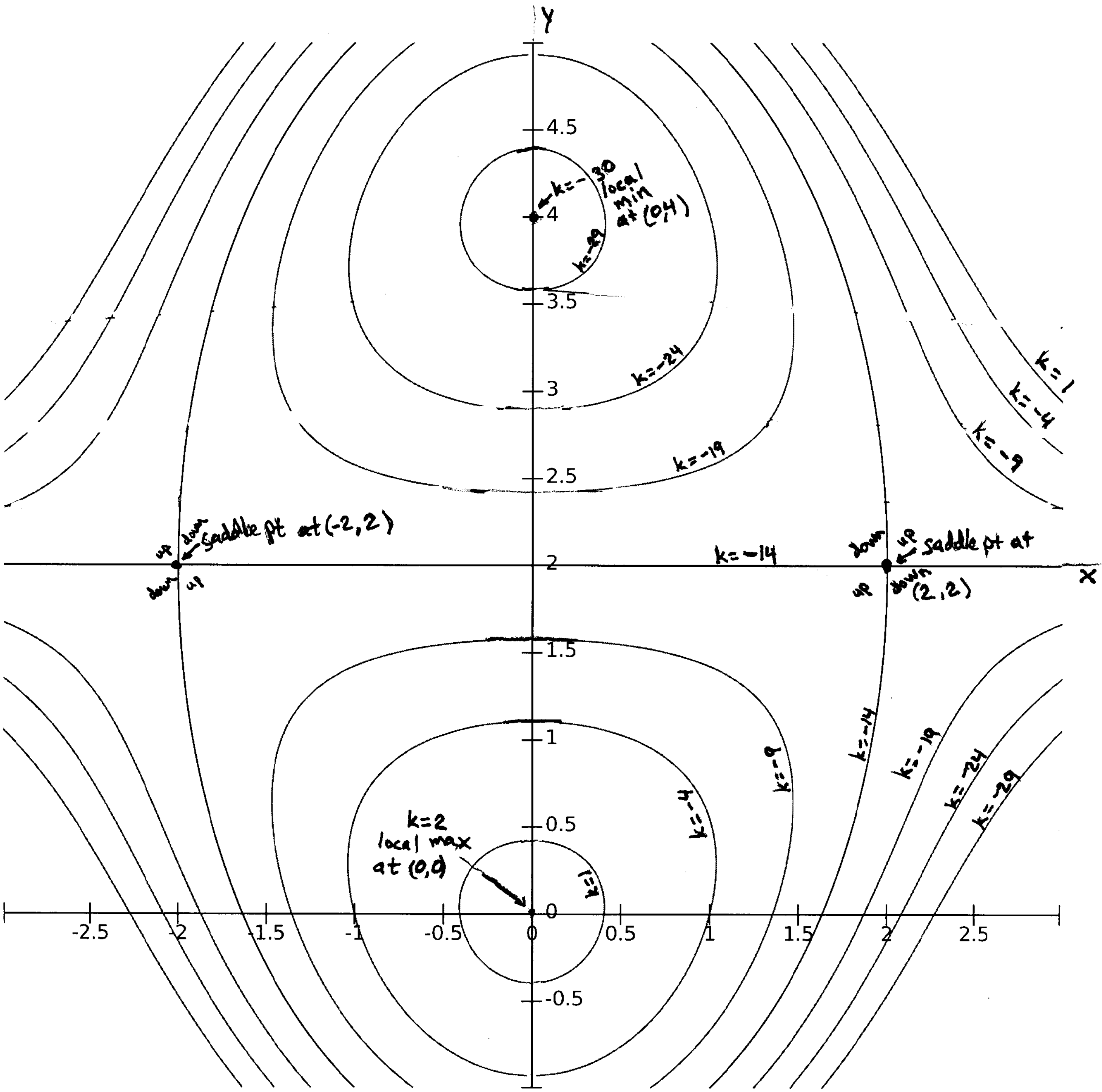
At $(0,4)$, $D = \begin{vmatrix} 12 & 0 \\ 0 & 12 \end{vmatrix} = 144 > 0$ and $f_{xx}(0,4) = 12 > 0$ so $\boxed{f(0,4) = -30 \text{ is a local minimum}}$

At $(2,2)$ $D = \begin{vmatrix} 0 & 12 \\ 12 & 0 \end{vmatrix} = -144 < 0$ so $\boxed{f(2,2) = -14 \text{ is a saddle point}}$

At $(-2,2)$ $D = \begin{vmatrix} 0 & -12 \\ -12 & 0 \end{vmatrix} = -144 < 0$ so $\boxed{f(-2,2) = -14 \text{ is a saddle point}}$.

Level curves of $f(x,y) = y^3 + 3x^2y - 6x^2 - 6y^2 + 2$

[§14.7, #9, p978 of Stewart Multivariable Calculus, 7th Ed.]



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