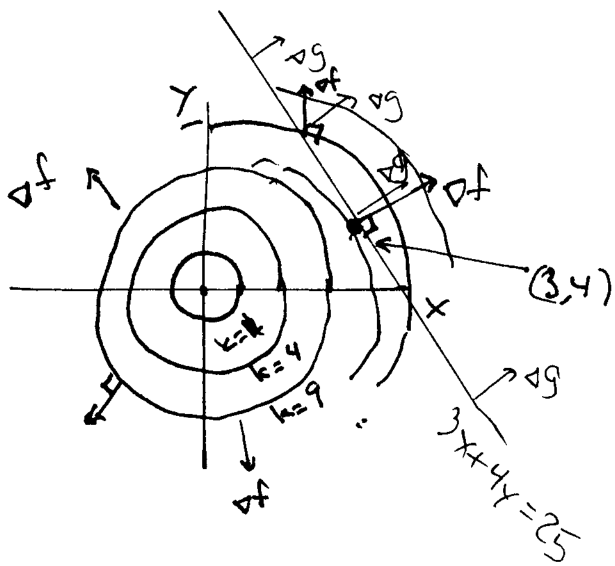
Postponed for now in §14.7: Absolute max/min of continuous functions on closed domains
[Reason: §14.8 will help solve these problems]

§14.8 Lagrange Multipliers, or "constrained optimization"

ex: Minimize the function $f(x,y) = x^2+y^2$ subject to the constraint $3x+4y=25$.



At the point where the minimum occurs, $\boxed{\nabla f(x,y) = \lambda \nabla g(x,y)}$

where $g(x,y)=k$ is the constraint equation.

We call $f(x,y)=x^2+y^2$ the "objective function"

Let $g(x,y) = 3x+4y$, so that $g(x,y)=25$ is the constraint equation.

$\nabla f(x,y) = \lambda \nabla g(x,y)$ becomes

$\langle 2x, 2y \rangle = \lambda \langle 3, 4 \rangle$ ← one vector can be broken out into two ordinary equations

$$\begin{cases} 2x = 3\lambda & \Rightarrow \lambda = \frac{2}{3}x \\ 2y = 4\lambda & \quad\; \lambda = \frac{1}{2}y \\ 3x+4y=25 \end{cases} \Rightarrow \frac{2}{3}x = \frac{1}{2}y \text{ or } y = \frac{4}{3}x$$

Sub into constraint eqn: $3x + 4\left(\frac{4}{3}x\right) = 25$

Multiply by 3:

$$9x + 16x = 75$$
$$25x = 75$$
$$x = 3$$

So $y = \frac{4}{3}(3) = 4$

The min value of $f(x,y)$ subject to the constraint is

$$f(3,4) = 3^2 + 4^2 = 25$$

Remark: Note that $\nabla f(3,4) = \langle 2(3), 2(4) \rangle = \langle 6, 8 \rangle$
while $\nabla g(3,4) = \langle 3, 4 \rangle$ so $\lambda = 2$ at $(3,4)$

14.8 9) Find max and min of $f(x,y,z) = xyz$
subject to $x^2 + 2y^2 + 3z^2 = 6$

Let $g(x,y,z) = x^2 + 2y^2 + 3z^2$

$\nabla f(x,y,z) = \lambda \nabla g(x,y,z)$ becomes

$$\langle yz, xz, xy \rangle = \lambda \langle 2x, 4y, 6z \rangle$$

$$\begin{cases} yz = 2\lambda x & (1) \\ xz = 4\lambda y & (2) \\ xy = 6\lambda z & (3) \\ x^2 + 2y^2 + 3z^2 = 6 & (4) \end{cases}$$

[The algebra of solving this nonlinear system of four equations and four variables will be completed in the notes.]

[Added after class ended:]

9 cont'd) Note: We may assume, since we're looking for places on the constraint surface where $f(x,y,z) = xyz$ is maximized and minimized, these will not occur where $xyz = 0$.

So, assume $x \neq 0$ and $y \neq 0$ and $z \neq 0$.

Now perform elimination, as follows:

$$\begin{array}{lrcl} 2y \cdot (1): & 2y^2 z &=& 4\lambda xy \\ -x \cdot (2): & -x^2 z &=& -4\lambda xy \\ \hline & 2y^2 z - x^2 z &=& 0 \end{array} \Rightarrow (2y^2 - x^2)z = 0$$

and since we're assuming $z \neq 0$, $2y^2 = x^2$. (equ 5)

$$\begin{array}{lrcl} 3z \cdot (1): & 3yz^2 &=& 6\lambda xz \\ -x \cdot (3): & -x^2 y &=& -6\lambda xz \\ \hline & 3yz^2 - x^2 y &=& 0 \end{array} \Rightarrow (3z^2 - x^2)y = 0$$

and since we assume $y \neq 0$, $3z^2 = x^2$. (equ 6)

Now, using equ (4): $x^2 + 2y^2 + 3z^2 = 6$, equ (5): $2y^2 = x^2$ and equ (6): $3z^2 = x^2$, we get $x^2 + x^2 + x^2 = 6 \Rightarrow 3x^2 = 6$
$\Rightarrow x^2 = 2 \Rightarrow x = \pm\sqrt{2}$

Since $x^2 = 2$, by (5) $2y^2 = 2 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$

Since $x^2 = 2$, by (6) $3z^2 = 2 \Rightarrow z^2 = \frac{2}{3} \Rightarrow z = \pm\sqrt{\frac{2}{3}}$

$\therefore$ $f(x,y,z) = xyz$, subject to the constraint $g(x,y,z) = 6$

is maximized at $\left(\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$,
$\left(-\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$

and minimized at $\left(-\sqrt{2}, 1, \sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2}, -1, \sqrt{\frac{2}{3}}\right)$,
$\left(\sqrt{2}, 1, -\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2}, -1, -\sqrt{\frac{2}{3}}\right)$

<u>Remark</u>: The system of equations {①, ②, ③, ④} has six more solutions, namely $\lambda = 0$ and

$$(x, y, z) = (\pm\sqrt{6}, 0, 0),$$
$$(0, \pm\sqrt{3}, 0), \text{ and}$$
$$(0, 0, \pm\sqrt{2}).$$

These six solutions act like saddle points of $f(x,y,z)$ subject to the constraint, and by the <u>Note</u> at the top of the previous page, f is neither maximized nor minimized there.