14.8 [one last topic in] Lagrange Multipliers

... maximizing $f(x,y,z)$ subject to

two constraints, $g(x,y,z)=k$ and $h(x,y,z)=c$.

Method: $\nabla f(x,y,z) = \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z)$

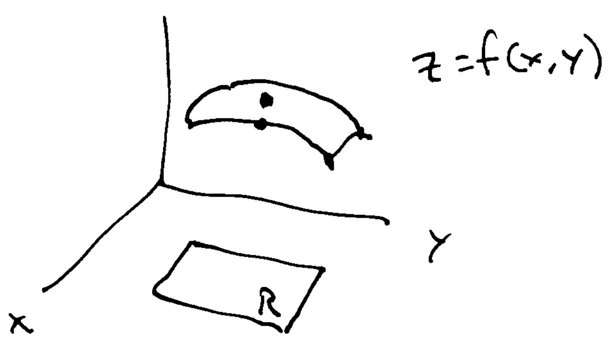
In effect, 3 eqns, 5 variables

$$\left.\begin{aligned} g(x,y,z) &= k \\ h(x,y,z) &= c \end{aligned}\right\}$$ 2 more equations. Total: 5 eqns, 5 variables.

Read the book; see Example 5 p.986. You're on your own.

Back to 14.7: Absolute max and min of a continuous $f(x,y)$ on a closed region.

Fact: If $f(x,y)$ is continuous on a closed, bounded region R then f attains its absolute maximum and minimum. These occur either where there is a critical point of f or on the boundary of R.



<u>Ex:</u> Find the <u>absolute</u> maximum and minimum of $f(x,y) = x^2+y^2-8x-6y$ on the closed, bounded region $R = \{(x,y) \mid x^2+y^2 \le 100\}$

<u>Step 1</u>: unconstrained optimization

Find critical points

$$0 = f_x(x,y) = 2x-8 = 2(x-4)$$
$$0 = f_y(x,y) = 2y-6 = 2(y-3)$$

$\Rightarrow (4,3)$ is the only C.P.

<u>Step 2</u>: <u>Constrained</u> optimization

Find the max and min of $f(x,y)$ on the circle $x^2+y^2=100$, that is, the boundary of R.

Use Lagrange multipliers. Let $g(x,y) = x^2+y^2$

$$\nabla f(x,y) = \lambda \nabla g(x,y)$$
$$\langle 2x-8, 2y-6 \rangle = \lambda \langle 2x, 2y \rangle$$

so

(1) $2x-8 = 2\lambda x$

(2) $2y-6 = 2\lambda y$

(3) $x^2+y^2 = 100$

Assume $x \neq 0$ and $y \neq 0$. Solve (1) and (2) for λ:

$$\left.\begin{aligned} \lambda &= \frac{2x-8}{2x} = \frac{2x}{2x} - \frac{8}{2x} = 1 - \frac{4}{x} \\ \lambda &= \frac{2y-6}{2y} = 1 - \frac{3}{y} \end{aligned}\right\} \quad 1 - \frac{4}{x} = 1 - \frac{3}{y}$$

$$\Rightarrow \frac{4}{x} = \frac{3}{y} \Rightarrow 4y = 3x \Rightarrow y = \frac{3}{4}x$$

Sub this into (3)

we get $x^2+\left(\frac{3}{4}x\right)^2 = 100$

$$\frac{16x^2}{16} + \frac{9x^2}{16} = 100$$

$$\frac{25x^2}{16} = 100 \Rightarrow x^2 = \frac{16 \cdot 100}{25} = 64$$

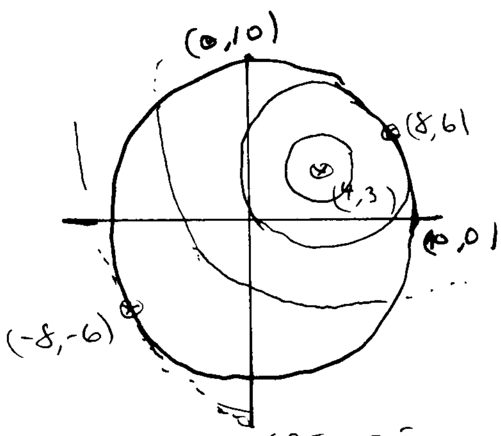
$$x = \pm 8 .$$

$$y = \frac{3}{4}(\pm 8) = \pm 6$$

So any (constrained) max or min happens at

$$(x,y) = (8,6) \text{ or } (x,y) = (-8,-6)$$

(x,y)	$f(x,y) = x^2+y^2-8x-6y$
$(4,3)$	$f(4,3) = 16+9-32-18 = -25$ = abs. min.
$(8,6)$	$f(8,6) = 64+36-64-36 = 0$
$(-8,-6)$	$f(-8,-6) = 64+36+64+36 = 200$ = abs. max.



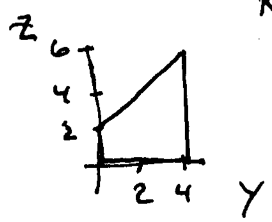
15.1 Double Integrals over Rectangles

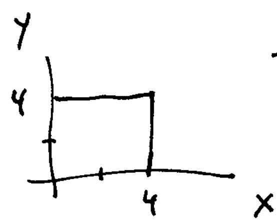
Idea of Defn:
$$\iint_R f(x,y)\,dA \approx \sum_i \sum_j f(x_{ij}, y_{ij})\,\Delta A$$

= sum of the volume of many skinny boxes
where ΔA = area of the base of the skinny box

ex: Evaluate the double integral by first identifying it as the volume of a solid

$$\iint_R (2+y)\,dA \quad , \quad R = \{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 4\}$$





Answer: $\iint_R (2+y)\,dA = 64$

Defn: The average value of $f(x,y)$ on a region R is

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA$$

So that $A(R) \cdot f_{ave} = \iint_R f(x,y)\,dA$ where $A(R)$ = area of R

In previous ex

$$f_{ave} = \frac{1}{A(R)} \iint_R f(x,y)\,dA = \frac{1}{4 \cdot 4} \cdot 64 = \frac{1}{16}(64) = 4$$

15.2 Iterated integrals

ex: Add these numbers:

1	2	4	8	15
2	4	8	16	30
3	6	12	24	45
4	8	16	32	60
10	20	40	80	150

ex: a) $\int_0^2 xy^2\,dx$

$$= y^2 \int_{x=0}^{x=2} x\,dx$$

$$= y^2 \left[\frac{x^2}{2} \right]_{x=0}^{2}$$

$$= y^2 \left[\frac{2^2}{2} - \frac{0^2}{2} \right]$$

$$= 2y^2$$

b) $\int_0^3 xy^2\,dy$

$$= x \int_{y=0}^{y=3} y^2\,dy$$

$$= x \left[\frac{y^3}{3} \right]_0^3$$

$$= x \left[\frac{3^3}{3} - \frac{0^3}{3} \right]$$

$$= 9x$$

[Notes completed after class ended.]

examples of an "iterated integral":

c) $$\int_0^3 \int_0^2 x y^2 \, dx \, dy = \int_0^3 2y^2 \, dy \quad \text{by part a)}$$

$$= \frac{2}{3} y^3 \Big|_0^3 = \frac{2}{3} \cdot 3^3 = 18$$

d) $$\int_0^2 \int_0^3 x y^2 \, dy \, dx = \int_0^2 9x \, dx \quad \text{by part b) above}$$

$$= \frac{9}{2} x^2 \Big|_0^2 = \frac{9}{2} \cdot 2^2 = 18$$

<u>Remark</u>: To the extent that <u>integrals</u> are like <u>sums</u>,

$\int_0^3 x y^2 \, dy$ from part b) is like calculating a <u>subtotal</u>, one for each y (column), by calculating a family of integrals (sums).

Then in part d) $\int_0^2 \int_0^3 x y^2 \, dy \, dx = \int_0^2 9x \, dx$ is like adding all the sub<u>totals</u> to get a <u>grand total</u>.