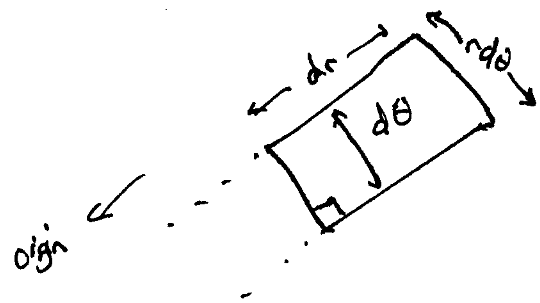
15.4 (A little more on) Polar coordinates

<u>Remark</u>: Why is dA = "element of area" = $r\,dr\,d\theta$?

dA is the amount of area in the plane swept out when you vary θ by a small amount, $d\theta$, and r by a small amount, dr.



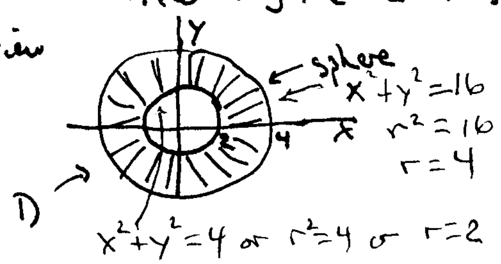
$d\theta$ is so small that the two rays are practically parallel lines, and the arcs of constant r are practically line segments

area $= dA =$ (length)(width) $= (dr)(r\,d\theta)$
$= r\,dr\,d\theta$

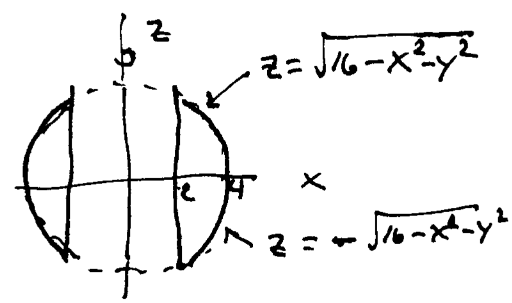
22) Use polar coordinates to find the volume <u>inside</u> the sphere $x^2+y^2+z^2=16$ and outside the cylinder $x^2+y^2=4$

view along the z-axis

Top view



Side View



volume $= 2$(volume of top half sphere) $= 2\iint_D \sqrt{16-x^2-y^2}\,dA = 2\iint_D \sqrt{16-(x^2+y^2)}\,dA$

$$= 2\int_0^{2\pi}\int_2^4 \sqrt{16-r^2}\;r\,dr\,d\theta = 2\int_0^{2\pi} d\theta \int_2^4 (16-r^2)^{1/2}\,r\,dr$$

$$= [\theta]_0^{2\pi} \int_2^4 (16-r^2)^{1/2}\cdot 2r\,dr$$

$u = 16 - r^2$

$du = -2r\,dr$

$-du = 2r\,dr$

$r = 2 \Leftrightarrow u = 12$

$r = 4 \Leftrightarrow u = 0$

$$= 2\pi \left[-\int_{12}^{0} u^{1/2}\,du \right]$$

$$= 2\pi \int_{0}^{12} u^{1/2}\,du$$

$$= 2\pi \cdot \frac{2}{3} u^{3/2} \Big|_{0}^{12}$$

$$= \frac{4\pi}{3} \cdot 12^{3/2} - 0 = \frac{4\pi}{3} \sqrt{12^3}$$

$$= \frac{4\pi}{3} \cdot 12\sqrt{12} = \frac{4\pi}{3} \cdot 12 \cdot 2\sqrt{3}$$

$$= 32\pi\sqrt{3}$$

Remark: why do we ever do double integrals in polar coordinates?

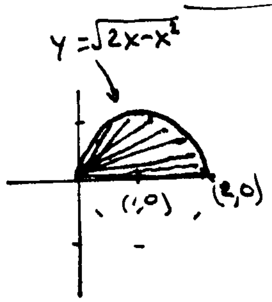
(1) To make the integrand look simpler (e.g. $\sqrt{16 - x^2 - y^2}$ vs. $\sqrt{16 - r^2} \cdot r$

(more important) (2) to make the limits simpler (e.g. $y = \sqrt{16 - x^2}$ vs. $r = 4$)

32) [Use of all three Ch 15 skills] Evaluate the iterated integral by converting to polar coordinates.

$$\int_0^2 \int_0^{\sqrt{2x - x^2}} \sqrt{x^2 + y^2}\,dy\,dx$$

Boundaries of D:



$y = \sqrt{2x - x^2} \Rightarrow y^2 = 2x - x^2 \Rightarrow x^2 + y^2 = 2x$

$r^2 = 2r\cos\theta \Rightarrow \boxed{r = 2\cos\theta}$

$y = 0 \Rightarrow r\sin\theta = 0 \Rightarrow \sin\theta = 0 \Rightarrow \boxed{\theta = 0}$

$x = 0 \Rightarrow r\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \boxed{\theta = \frac{\pi}{2}}$

$x = 2$

Note: $x^2 - 2x + 1 + y^2 = 0 + 1$

$(x-1)^2 + y^2 = 1$

Since $x^2+y^2=r^2$

15.4 32 cont'd)
$$\int_0^{\pi/2}\int_0^{2\cos\theta}\sqrt{r^2}\cdot r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta$$

$$= \int_0^{\pi/2} \frac{r^3}{3}\Big|_0^{2\cos\theta} d\theta = \int_0^{\pi/2} \frac{8}{3}\cos^3\theta\,d\theta$$

$$= \int_0^{\pi/2} \frac{8}{3}(1-\sin^2\theta)\cos\theta\,d\theta$$

$$= \frac{8}{3}\int_0^{\pi/2}\cos\theta\,d\theta - \frac{8}{3}\int_0^{\pi/2}\sin^2\theta\cos\theta\,d\theta$$

"in-line substitution"

$$= \left[\frac{8}{3}\sin\theta\right]_0^{\pi/2} - \frac{8}{3}\int_0^{\pi/2}\sin^2\theta\,d(\sin\theta)$$

$$= \left[\frac{8}{3}\sin\theta\right]_0^{\pi/2} - \frac{8}{9}\left[\sin^3\theta\right]_0^{\pi/2}$$

$$= \frac{8}{3} - \frac{8}{9} = \frac{24-8}{9} = \boxed{\frac{16}{9}}$$