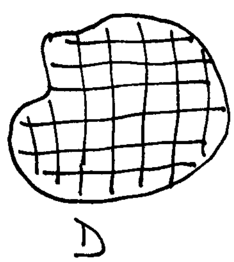
Math 281 - Mon 25 Apr, 2016

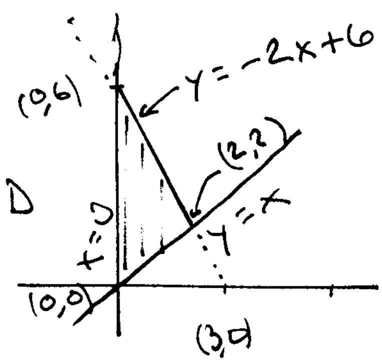
15.5 Mass and center of mass of a lamina

ex: Suppose you know the mass density, $\rho(x,y)$, in grams/cm^2 of a piece of sheet metal, and you know the shape of the sheet metal, what is the mass?

$$\text{mass} = m = \iint_D \rho(x,y)\, dA \simeq \sum_x \sum_y \rho(x,y)\Delta A$$



6) Find the mass of the lamina D where $\rho(x,y) = x^2$ g/cm^2 and the boundaries of D are the lines $x=0$, $y=x$, and $2x+y=6$.



$$m = \int_0^2 \int_x^{-2x+6} x^2\, dy\, dx = \int_0^2 x^2 y \Big]_x^{-2x+6} dx$$

$$= \int_0^2 \left[x^2(-2x+6) - x^3\right] dx = \int_0^2 (-2x^3+6x^2-x^3)\,dx$$

$$= \int_0^2 (-3x^3+6x^2)\,dx = \left[-\frac{3}{4}x^4 + 2x^3\right]_0^2$$

$$= -\frac{3}{4}\cdot 2^4 + 2\cdot 2^3 = -12+16 = \boxed{4 \text{ grams}}$$

Aside: What is the average mass density of this lamina?

Answer = $\frac{\text{mass}}{\text{area}} = \frac{1}{\text{area}} \iint_D \rho(x,y)\,dA = \frac{1}{6\text{ cm}^2}(4 \text{ grams}) = \frac{2}{3}$ g/cm^2

<u>Defn</u>: The <u>center of mass</u> of lamina, D, with mass density $\rho(x,y)$ is $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA$$ = "weighted" average of the x-coordinates of each molecule in D.

$$\bar{y} = \frac{1}{m} \iint_D y\,\rho(x,y)\,dA$$

units? $$\underbrace{\bar{x}}\ \text{cm} = \underbrace{\left(\frac{1}{g}\right)}_{1/m} \cdot \underbrace{(\text{cm})}_{x} \underbrace{\left(g/\text{cm}^2\right)}_{\rho(x,y)} \underbrace{(\text{cm}^2)}_{dA}$$

<u>Defn</u>: Special case of $\rho(x,y) = 1$

$$m = \text{Area} = \iint_D dA \qquad (\bar{x}, \bar{y}) = \text{centroid}.$$

6) [cont'd] Find the <u>center of mass</u> of D. <u>Recall</u>: m = 4 grams

$$\bar{x} = \frac{1}{m} \iint_D x\,\rho(x,y)\,dA = \frac{1}{4} \int_0^2 \int_{*}^{-2x+6} x \cdot x^2\,dy\,dx$$

$$= \frac{1}{4} \int_0^2 x^3 y \Big]_x^{-2x+6} dx = \frac{1}{4} \int_0^2 \left[x^3(-2x+6) - x^4\right] dx$$

$$= \frac{1}{4} \int_0^2 \left(-3x^4 + 6x^3\right) dx = \frac{1}{4}\left[-\frac{3}{5}x^5 + \frac{3}{2}x^4\right]_0^2$$

$$= \frac{1}{4}\left[-\frac{3}{5}\cdot 2^5 + \frac{3}{2}\cdot 2^4\right] = \frac{2^3}{4}\left[-\frac{3}{5}\cdot 2^2 + 3\right] = 2\left[\frac{-12}{5} + \frac{15}{5}\right]$$

$$= \frac{6}{5} = 1.2 \text{ cm}$$

6 cont'd)

$$\bar{y} = \frac{1}{m} \iint_D y\,\rho(x,y)\,dA = \frac{1}{4} \int_0^2 \int_x^{-2x+6} y \cdot x^2 \, dy\,dx$$

$$= \frac{1}{4} \int_0^2 \frac{1}{2} x^2 y^2 \Big]_x^{-2x+6} dx = \frac{1}{8} \int_0^2 \left[(-2x+6)^2 x^2 - x^2 \cdot x^2\right] dx$$

$$= \frac{1}{8} \int_0^2 \left[(4x^2 - 24x + 36)x^2 - x^4 \right] dx$$

$$= \frac{1}{8} \int_0^2 (4x^4 - 24x^3 + 36x^2 - x^4)\, dx = \frac{1}{8} \int_0^2 (3x^4 - 24x^3 + 36x^2)\, dx$$

$$= \frac{3}{8} \left[\frac{x^5}{5} - 2x^4 + 4x^3 \right]_0^2$$

$$= \frac{3}{8} \left[\frac{2^5}{5} - 2^5 + 2^5 \right] = \frac{12}{5} = 2.4 \text{ cm}$$

$$\therefore \quad (\bar{x}, \bar{y}) = (1.2 \text{ cm}, 2.4 \text{ cm})$$

15.6 Surface Area ...

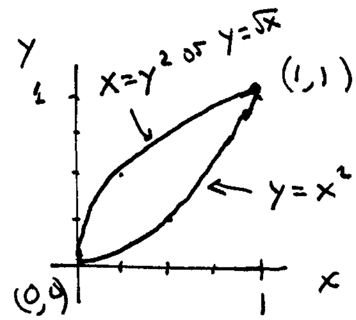
will be skipped, because we will cover a more general concept in section 16.7: Surface Integrals.

15.7 Triple Integrals

[See the textbook for the <u>definition</u> of triple integrals over a solid region E, as well as the triple integral version of <u>Fubini's Theorem</u>, which tells us how to set up an <u>iterated</u> <u>integral</u> to calculate a triple integral.]

14) Find $\iiint_E xy\,dV$ where E is the solid region with boundaries $y=x^2$, $x=y^2$, $z=0$, and $z=x+y$.

Top view of E



$$\iiint_E xy\,dV = \int_0^1 \int_{x^2}^{\sqrt{x}} \int_0^{x+y} xy\,dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} xyz\Big]_0^{x+y} dy\,dx$$

$$= \int_0^1 \int_{x^2}^{\sqrt{x}} [xy(x+y) - xy\cdot(0)]\,dy\,dx = \int_0^1 \int_{x^2}^{\sqrt{x}} (x^2y + xy^2)\,dy\,dx$$

$$= \int_0^1 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_{x^2}^{\sqrt{x}} dx = \int_0^1 \left[\left(\frac{1}{2}x^2\cdot x + \frac{1}{3}x\cdot x^{3/2}\right) - \left(\frac{1}{2}x^2\cdot(x^2)^2 + \frac{1}{3}x\cdot(x^2)^3\right)\right] dx$$

$$= \int_0^1 \left(\frac{1}{2}x^3 + \frac{1}{3}x^{5/2} - \frac{1}{2}x^6 - \frac{1}{3}x^7\right) dx$$

$$= \left[\frac{1}{8}x^4 + \frac{2}{21}x^{7/2} - \frac{1}{14}x^7 - \frac{1}{24}x^8\right]_0^1 = \frac{1}{8} + \frac{2}{21} - \frac{1}{14} - \frac{1}{24}$$

$$= \boxed{\frac{3}{28}}$$