Some very informal Notes on Linear Transformations [see any Linear Algebra textbook]

Remark: Linear transformations are functions from $\mathbb{R}^n$ to $\mathbb{R}^m$ which preserves the vector structure of $\mathbb{R}^n$ and $\mathbb{R}^m$.

That is a linear transformation $T:\mathbb{R}^n \to \mathbb{R}^m$ satisfies

i) $T(\vec{v}+\vec{w}) = T(\vec{v})+T(\vec{w})$ for all vectors $\vec{v}$ and $\vec{w}$ in $\mathbb{R}^n$

and ii) $T(c\vec{v}) = c\,T(\vec{v})$ for all vectors $\vec{v}\in\mathbb{R}^n$ and scalars $c\in\mathbb{R}$.

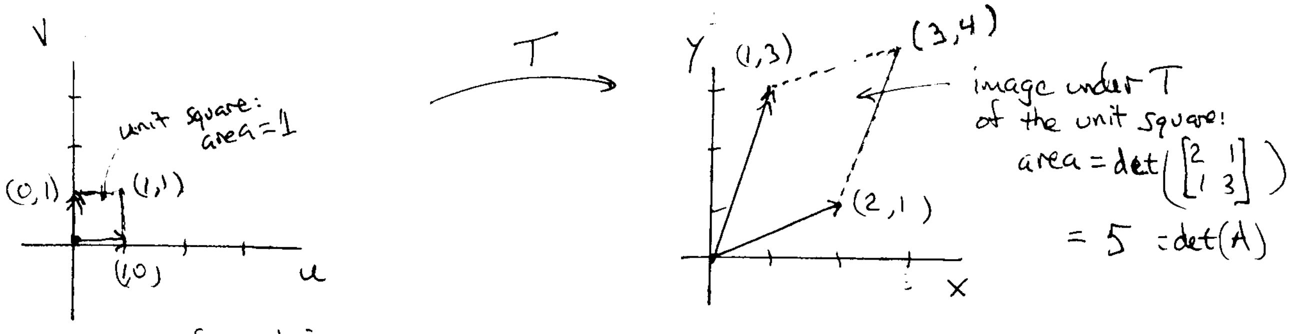
One can show that a given linear transformation T can be represented by multiplication on the left by a certain $m\times n$ matrix.

example: If $T:\mathbb{R}^2\to\mathbb{R}^2$ has the property that

$$T(1,0) = (2,1) \quad \text{and} \quad T(0,1) = (1,3)$$

then $T(u,v) = (x,y)$ where the relation between (u,v) in the domain of T and (x,y) in the range of T is given by

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = A\begin{bmatrix} u \\ v \end{bmatrix} \quad \text{where } A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$$



Facts about linear transformations:

1. Under T, the image of a line through the origin is a line through the origin.
2. Parallel lines map to parallel lines, so parallelograms map to parallelograms. Also parallel planes map to parallel planes, etc.

Facts about linear transformations [cont'd]

(3) If $n=m$, $T: \mathbb{R}^n \to \mathbb{R}^n$ is represented by an $n \times n$ matrix A

and $\det(A) = \pm$ area (or volume or hypervolume) of the image of the unit square (or cube or hypercube),

'+' if T preserves orientation (maps counterclockwise-oriented parallelograms to counter-clockwise oriented parallelograms),

'−' if T reverses orientation.

If $\det(A) = 0$, the image of the unit square is a degenerate parallelogram having zero area, which means that T is not one-to-one.

(4) If $T: \mathbb{R}^n \to \mathbb{R}^n$ is one-to-one (i.e. $\det(A) \neq 0$) then T^{-1} is also a linear transformation and is represented by the $n \times n$ matrix A^{-1}.

(5) If $T: \mathbb{R}^n \to \mathbb{R}^m$ is represented by the $m \times n$ matrix A

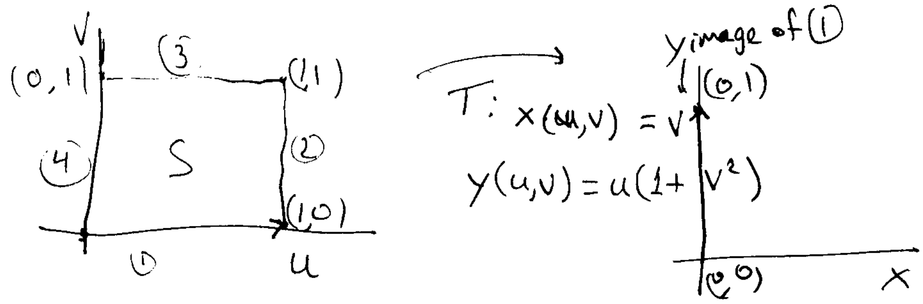
and $S: \mathbb{R}^m \to \mathbb{R}^p$ is represented by the $p \times m$ matrix B

then $S \circ T: \mathbb{R}^n \to \mathbb{R}^p$ is a linear transformation represented by the $p \times n$ matrix BA.

and $\det(BA) = \det(B) \cdot \det(A)$.

15.10 Double (and Triple) integrals under a change of variables

8) $T: \mathbb{R}^2 \to \mathbb{R}^2$ will be a non-linear transformation



T: $x(u,v) = v$
$y(u,v) = u(1+v^2)$

For example $T(0,0) = (0,0)$
$T(1,0) = (0, 1\cdot(1+0^2)) = (0,1)$

~~$S = \{(u,v)\}$~~ S is bounded by $u=0,\ u=1,\ v=0,\ v=1$.

Image of line 1 under T? On 1: u = variable, $v=0$

So $x[=v]=0$ and $y[=u(1+v^2)]=u$

parametric curve: $x(u)=0$ $\quad 0 \le u \le 1$
$y(u) = u$

Image of 2? On 2 $u=1$, $0 \le v \le 1$

$x(1,v) = v$
$y(1,v) = 1\cdot(1+v^2) = 1+v^2$

we can eliminate the parameter: $y = 1+x^2$

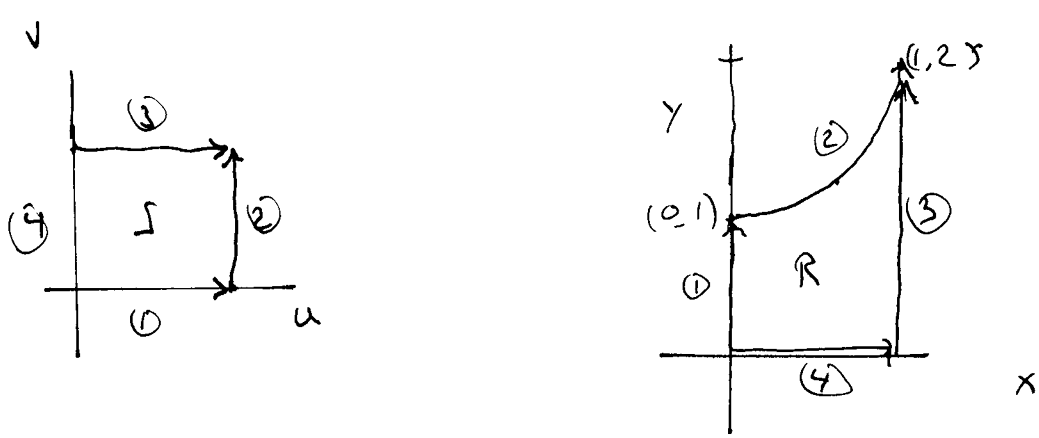


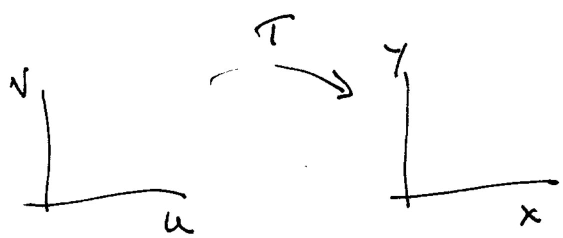
Image of (3)? on (3): $v = 1,\ 0 \le u \le 1$

$$x(u,v) = x(u,1) = 1$$
$$y(u,v) = y(u,1) = u(1+1^2) = 2u$$

Image of (4)? on (4): $u = 0,\ 0 \le v \le 1$

$$x(0,v) = v$$
$$y(0,v) = 0(1+v^2) = 0$$

<u>Defn</u>: The <u>Jacobian</u> of a transformation T given by $\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$



is denoted $\dfrac{\partial(x,y)}{\partial(u,v)}$

and is calculated as:

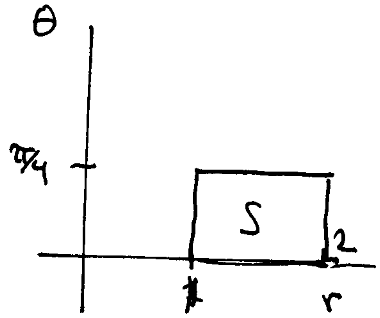
$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

ex: The Jacobian for T in problem 8 where $\begin{cases} x = v \\ y = u(1+v^2) \end{cases}$

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

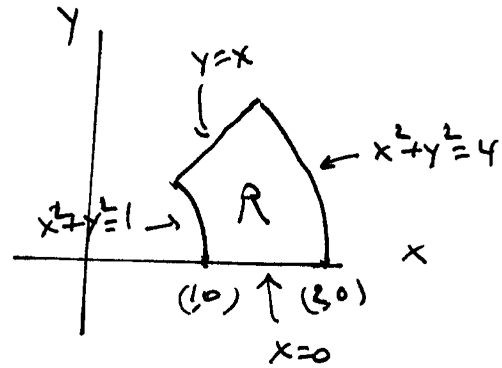
$$= \begin{vmatrix} 0 & 1 \\ 1+v^2 & 2uv \end{vmatrix} = -(1+v^2) = \text{distortion of area under } T$$

Polar coordinates as a change of variables



$$\xrightarrow{T}$$

$$\begin{cases} x(r,\theta) = r\cos\theta \\ y(r,\theta) = r\sin\theta \end{cases}$$



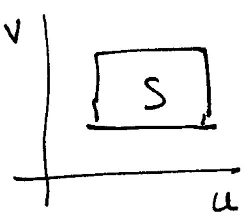
$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (\cos\theta)(r\cos\theta) - (\sin\theta)(-r\sin\theta)$$

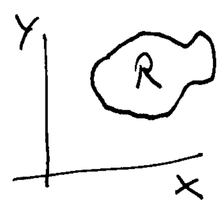
$$= r\cos^2\theta + r\sin^2\theta$$

$$= r(\cos^2\theta + \sin^2\theta) = \boxed{r}$$

Theorem $\boxed{9}$:
p 1068



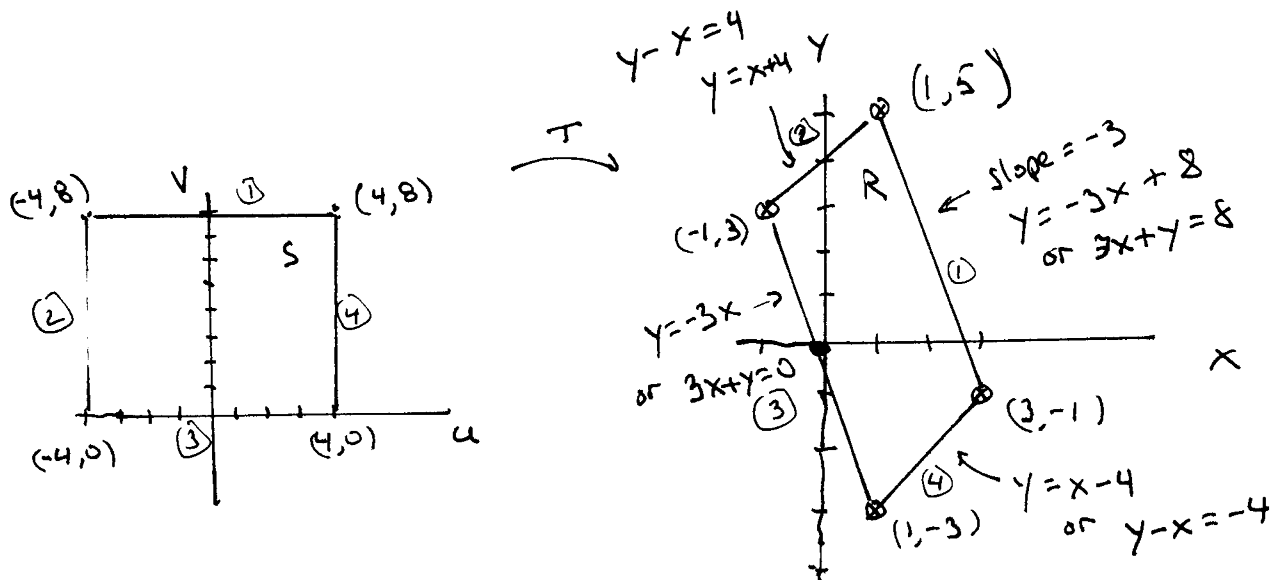
$$\xrightarrow{T} \quad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$



$$\boxed{\iint_R f(x,y)\,dx\,dy = \iint_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

16) $\iint_R (4x+8y)\,dA$

R is a parallelogram with vertices $(-1,3)$, $(1,-3)$ $(3,-1)$, $(1,5)$



T: $x = \frac{1}{4}(u+v)$

$y = \frac{1}{4}(v-3u)$

so $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{4} & \frac{1}{4} \\ -\frac{3}{4} & \frac{1}{4} \end{vmatrix} = \frac{1}{4}$

Boundaries of S? ① $3x+y=8$ becomes $3\cdot\frac{1}{4}(u+v) + \frac{1}{4}(v-3u) = 8$

or $\cancel{\frac{3}{4}u} + \frac{3}{4}v + \frac{1}{4}v - \cancel{\frac{3}{4}u} = 8$

or $v = 8$

③ $3x+y=0$ becomes $3\cdot\frac{1}{4}(u+v) + \frac{1}{4}(v-3u) = 0$

or $v = 0$

② $y - x = 4$ becomes $\frac{1}{4}(v-3u) - \frac{1}{4}(u+v) = 4$

or $\frac{1}{4}v - \frac{3}{4}u - \frac{1}{4}u - \frac{1}{4}v = 4$

or $-u = 4$ or $u = -4$

④ $y - x = -4$ becomes $\frac{1}{4}(v-3u) - \frac{1}{4}(u+v) = -4$

or $u = 4$

(6 cont'd) The integrand $f(x,y)$ of the integral we want to calculate becomes, under the change of variables:

$$4x+8y = 4\cdot\tfrac{1}{4}(u+v) + 8\cdot\tfrac{1}{4}(v-3u)$$
$$= u+v + 2v - 6u$$
$$= -5u + 3v$$

We're now ready to calculate the integral:

$$\iint_R (4x+8y)\,dx\,dy = \iint_S (-5u+3v)\,\frac{\partial(x,y)}{\partial(u,v)}\,du\,dv$$
$$= \int_0^8\int_{-4}^4 (-5u+3v)\cdot\tfrac{1}{4}\,du\,dv$$
$$= -\tfrac{5}{4}\int_0^8\int_{-4}^4 u\,du\,dv + \tfrac{3}{4}\int_0^8\int_{-4}^4 v\,du\,dv$$
$$= -\tfrac{5}{4}\cdot\underbrace{\cancel{\int_{-4}^4 u\,du}}_{0}\cdot\int_0^8 dv + \tfrac{3}{4}\cdot\int_{-4}^4 du\cdot\int_0^8 v\,dv$$
$$= \tfrac{3}{4}\Big[u\Big]_{-4}^{4}\Big[\tfrac{v^2}{2}\Big]_0^8$$
$$= \tfrac{3}{4}\,[4-(-4)]\,[32-0]$$
$$= \tfrac{3}{4}\cdot 8\cdot 32 = \boxed{192}$$

Remark: In this example, the transformation T is a linear transformation represented by $\begin{bmatrix}x\\y\end{bmatrix} = A\begin{bmatrix}u\\v\end{bmatrix} = \begin{bmatrix}\frac14 & \frac14\\ -\frac34 & \frac14\end{bmatrix}\begin{bmatrix}u\\v\end{bmatrix}$ and $\det(A) = \frac{\partial(x,y)}{\partial(u,v)} = \frac14$ shows T diminishes area by a factor of $\frac14$.