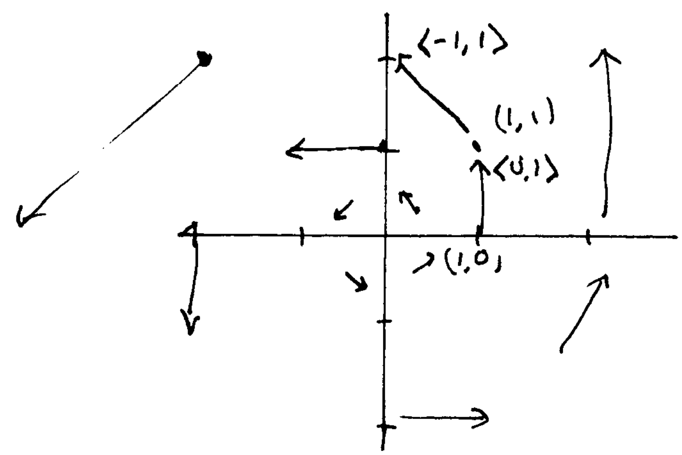
16.1 Vector fields

<u>example</u>: $\vec{F}(x,y) = \langle -y, x \rangle$

(x,y)	$\langle -y, x \rangle$
$(1,0)$	$\langle 0, 1 \rangle$
$(0,1)$	$\langle -1, 0 \rangle$
$(1,1)$	$\langle -1, 1 \rangle$
$(1,2)$	$\langle -2, 1 \rangle$
$(0,0)$	$\langle 0, 0 \rangle$



<u>Remark</u>. Every gradient vector field is a vector field.

<u>ex</u>. $f(x,y) = x^2 y$

$$\vec{F}(x,y) = \nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \langle 2xy, x^2 \rangle$$
$$= \langle P(x,y), Q(x,y) \rangle$$

where $P(x,y) = 2xy$,
and $Q(x,y) = x^2$.

<u>Observe</u>: In this case that $\vec{F}(x,y) = \nabla f(x,y)$ = a gradient
we have that $\frac{\partial P}{\partial y} = f_{xy}(x,y) = 2x = f_{yx}(x,y) = \frac{\partial Q}{\partial x}$.

equal by Clairaut's Theorem

That is, $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$ ← <u>Test</u> for conservative vector fields

So, is <u>every</u> vector field the gradient $\nabla f(x,y)$ for some real-valued function $f(x,y)$?

<u>Nope.</u> In the first example $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle$

$$= \langle -y, x \rangle$$

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[-y] = -1 \neq 1 = \frac{\partial}{\partial x}[x] = \frac{\partial Q}{\partial x}.$$

<u>Defn</u>: A vector field $\vec{F}(x,y)$ is a <u>conservative vector field</u> if $\vec{F}(x,y) = \nabla f(x,y)$ for some real-valued function (or "scalar field") $f(x,y)$.

If so, $f(x,y)$ is called a <u>potential function</u> for $\vec{F}$.

<u>ex</u>: Is the following vector field conservative?

$$F(x,y) = \langle 3x^2 - 2y^2, 4xy + 3 \rangle$$
$$= \langle P(x,y), Q(x,y) \rangle$$

[§16.3 #6]

$$\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}[4xy+3] = 4y$$
$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}[3x^2 - 2y^2] = -4y$$

Not equal, so not conservative; that is, $F \neq \nabla f(x,y)$

Equivalent form of the conservative test:

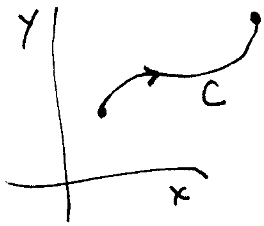
$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial}{\partial x} - \frac{\partial P}{\partial y} = 0 \ ?$$

16.2 Line integrals

Remark : There are two types of line integrals

(1) Line integral of a real-valued function

$$\int_C f(x,y)\,ds.$$



Intuitively, this is "weighted arc length"
where ds = element of arc length (in cm)
and $f(x,y)$ = mass density (g/cm) of a wire.

Then $\int_C f\,(g/cm) \cdot ds\,(cm) = \text{mass}\,(g)$

(2) Line integral of a vector field $\vec{F} = \langle P(x,y), Q(x,y) \rangle$

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int \vec{F} \cdot d\vec{r}$$

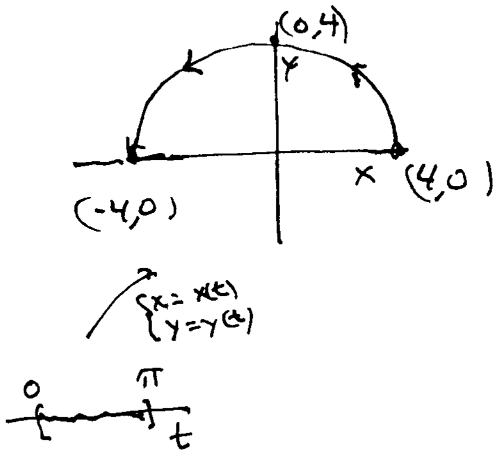
In physics, this is how we calculate work.

Defn : For a parametric curve $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $a \le t \le b$ [Equivalent: $\vec{r}(t) = \langle x(t), y(t) \rangle$]

and a real-valued function $f(x,y)$

$$\boxed{\begin{aligned} \int_C f(x,y)\,ds &= \int_{t=a}^{b} f(x(t),y(t))\,\frac{ds}{dt}\,dt \\ &= \int_{t=a}^{b} f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \end{aligned}}$$

ex: Evaluate $\int_C y\,ds$ where $C: \begin{cases} x = 4\cos t \\ y = 4\sin t \end{cases}$ $0 \le t \le \pi$



$$\frac{dx}{dt} = -4\sin t$$

$$\frac{dy}{dt} = 4\cos t$$

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$$

$$\int_C y\,ds = \int_0^\pi \underbrace{(4\sin t)}_{y} \cdot \underbrace{4}_{\frac{ds}{dt}}\,dt = 16\int_0^\pi \sin t\,dt$$

$$= -16\cos t\Big]_0^\pi$$

$$= -16[\cos\pi - \cos 0]$$

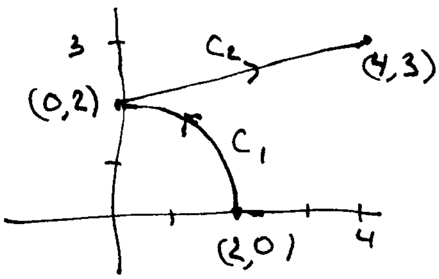
$$= -16[-1-1] = \boxed{32}$$

ex: Evaluate $\int_C x\,ds$ for the same C. <u>Answer</u>: 0

[exercise] [Note: Symmetry].

8) Evaluate $\int_C x^2\,dx + y^2\,dy$

where C consists of the arc of the circle $x^2+y^2=4$ from $(2,0)$ to $(0,2)$ followed by the line segment from $(0,2)$ to $(4,3)$.



For C_1, take $\begin{cases} x = 2\cos t \\ y = 2\sin t \\ 0 \le t \le \pi/2 \end{cases}$ to be the parametrization,

so $dx = -2\sin t\,dt$
$dy = 2\cos t\,dt$

$$\int_{C_1} x^2\,dx + y^2\,dy = \int_0^{\pi/2} \left[(2\cos t)^2(-2\sin t) + (2\sin t)^2(2\cos t)\right] dt$$

$$= -8\int_0^{\pi/2} \cos^2 t \sin t\,dt + 8\int \sin^2 t \cos t\,dt$$

$$= 8\int_0^{\pi/2} \cos^2 t\, d(\cos t) + 8\int \sin^2 t\, d(\sin t)$$

$$= \left[\frac{8}{3}\cos^3 t + \frac{8}{3}\sin^3 t\right]_0^{\pi/2}$$

$$= \frac{8}{3} - \frac{8}{3} = 0$$

(to be continued)

[Added after the end of class...]

8 cont'd)

For C_2, we can parametrize the line from (0,2) to (4,3) by letting x itself be the parameter.

$$\begin{cases} x = t \\ y = \frac{1}{4}t + 2 \end{cases} \quad 0 \le t \le 4$$

so that when $t=0$, $(x,y) = (0,2)$ and when $t=4$, $(x,y) = (4,3)$.

Then $\begin{cases} dx = dt \\ dy = \frac{1}{4}\,dt \end{cases}$ and

$$\int_{C_2} x^2\,dx + y^2\,dy = \int_0^4 \left([x(t)]^2 \frac{dx}{dt} + [y(t)]^2 \frac{dy}{dt} \right) dt$$

$$= \int_0^4 \left[(t^2)(1) + \left(\frac{t}{4}+2\right)^2 \frac{1}{4} \right] dt = \int_0^4 \left(t^2 + \frac{1}{64}t^2 + \frac{1}{4}t + 1 \right) dt$$

$$= \int_0^4 \left(\frac{65}{64}t^2 + \frac{1}{4}t + 1 \right) dt = \left[\frac{65}{192}t^3 + \frac{1}{8}t^2 + t \right]_0^4$$

$$= \frac{65}{3} + 2 + 4 = \frac{83}{3} = 27.6\overline{6}$$

$$\therefore \int_C x^2\,dx + y^2\,dy = \int_{C_1} x^2\,dx + y^2\,dy + \int_{C_2} x^2\,dx + y^2\,dy = 0 + \frac{83}{3} = \boxed{\frac{83}{3}}$$

<u>Remarks</u>: (1) The result of the calculation of a line integral of a real-valued function $\int_C f(x,y)\,ds$, does not depend on the particular parametrization of C.

(2) The line integral of a vector field, $\int_C P\,dx + Q\,dy$, does not depend on the parametrization of the curve C, <u>except</u> that reversing the <u>orientation</u> of C has the effect of toggling the <u>sign</u> of $\int_C P\,dx + Q\,dy$. That is, if $-C$ denotes the curve C with <u>reversed</u> orientation, $\int_{-C} P\,dx + Q\,dy = -\int_C P\,dx + Q\,dy$.