Math 281 - Monday 9 May 2016

16.5 Curl and Divergence (and Gradient)

Remark These are all (sort of) derivatives.

ex: (gradient) $f(x,y,z) = x^2 \sin y + z^3$

$$\vec{F}(x,y,z) = \nabla f(x,y,z) = \operatorname{grad} f(x,y,z)$$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle (x^2 \sin y + z^3)$$

$$= \left\langle \frac{\partial}{\partial x}[x^2 \sin y + z^3], \frac{\partial}{\partial y}[x^2 \sin y + z^3], \frac{\partial}{\partial z}[x^2 \sin y + z^3] \right\rangle$$

$$= \langle 2x \sin y, x^2 \cos y, 3z^2 \rangle$$

NOTE: By definition, $\vec{F}$ is a conservative vector field with potential function f.

Defn: For a vector field $\vec{F}(x,y,z) = \langle P(x,y,z), Q(x,y,z), R(x,y,z) \rangle$ we define curl $\vec{F} = \nabla \times \vec{F}$ is as

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\hat{\imath} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\hat{\jmath} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\hat{k}$$

Note: $\nabla \times \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \times \langle P, Q, R \rangle$

2) For $\vec{F}(x,y,z) = \langle xy^2z^3,\ x^3yz^2,\ x^2y^3z \rangle$
find $\text{curl}\,\vec{F} = \nabla \times F$.

$$\text{curl}\,\vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^3 & x^3yz^2 & x^2y^3z \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y} x^2y^3z - \frac{\partial}{\partial z} x^3yz^2\right)\hat{\imath}$$
$$+ \left(\frac{\partial}{\partial z} xy^2z^3 - \frac{\partial}{\partial x} x^2y^3z\right)\hat{\jmath}$$
$$+ \left(\frac{\partial}{\partial x} x^3yz^2 - \frac{\partial}{\partial y} xy^2z^3\right)\hat{k}$$

$$= \left(3x^2y^2z - 2x^3yz\right)\hat{\imath}$$
$$+ \left(3xy^2z^2 - 2xy^3z\right)\hat{\jmath}$$
$$+ \left(3x^2yz^2 - 2xyz^3\right)\hat{k} = \vec{G}(x,y,z)$$

Remark: curl $\vec{F}$ is itself a vector field. In particular

$$\vec{G}(1,1,1) = \text{curl}\,F(1,1,1) = \langle 3-2,\ 3-2,\ 3-2 \rangle$$
$$= \langle 1, 1, 1 \rangle$$

Defn: For a vector field $\vec{F}(x,y,z) = \langle P, Q, R \rangle$ we define the divergence of $\vec{F}$ as follows:

$$\text{div}\,\vec{F}(x,y,z) = \nabla \cdot \vec{F}$$
$$= \frac{\partial}{\partial x} P(x,y,z) + \frac{\partial}{\partial y} Q(x,y,z) + \frac{\partial}{\partial z} R(x,y,z)$$

NOTE: $\nabla \cdot \vec{F} = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \rangle \cdot \langle P, Q, R \rangle$

Remark: (1) If $f(x,y,z)$ is a real-valued function, ∇f is a vector field

(2) If $\vec{F}(x,y,z)$ is a vector field, curl $\vec{F}$ is another vector field

(3) If $\vec{F}(x,y,z)$ is a vector field, div $\vec{F}$ is a real-valued function

ex: For $\vec{F}(x,y,z) = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x} xy^2z^3 + \frac{\partial}{\partial y} x^3yz^2 + \frac{\partial}{\partial z} x^2y^3z$$
$$= y^2z^3 + x^3z^2 + x^2y^3$$

Theorem: (1) For any real-valued function $f(x,y,z)$

$$\text{curl}(\nabla f) = \vec{0}$$

(2) For any vector field $\vec{F}(x,y,z)$

$$\text{div curl}\,\vec{F} = 0$$

Remark: So, is $\vec{F}(x,y,z) = \langle xy^2z^3, x^3yz^2, x^2y^3z \rangle$ conservative? No, because curl $\vec{F} \neq \vec{0}$.
Had $\vec{F} = \nabla f$ for some $f(x,y,z)$, curl $\vec{F} = \text{curl}(\nabla f) = \vec{0}$.

16.2 Line integrals

Two types: (1) $\int_C f(x,y)\,ds$

(2) $\int_C P(x,y)\,dx + Q(x,y)\,dy$

$$= \int_C \langle P(x,y), Q(x,y)\rangle \cdot \langle dx, dy\rangle$$

$$= \int_C \vec{F}(x,y) \cdot d\vec{r}$$

<u>How to calculate these</u> given a parametrization of C

$C: \begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $a \le t \le b$ OR $\langle x,y\rangle = \vec{r}(t)$, $a \le t \le b$

<u>HINT</u>: Let the notation guide you.

<u>ex</u>: 2) $\int_C xy\,ds$ where $C: \begin{cases} x = t^2 \\ y = 2t \end{cases}$ $0 \le t \le 1$

[type(1)]

$$= \int_0^1 [x(t)]\,[y(t)]\left(\frac{ds}{dt}\right)dt$$

<u>Recall</u>: $\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$

$= \sqrt{(2t)^2 + 2^2}$

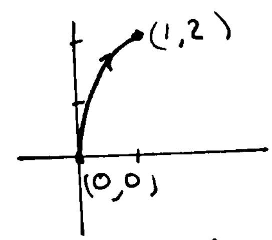
$= \sqrt{4t^2+4} = 2\sqrt{t^2+1}$

$$= \int_0^1 (t^2)(2t)\cdot 2\sqrt{t^2+1}\,dt$$

$$= \int_0^1 4t^3\sqrt{t^2+1}\,dt \leftarrow A$$

<u>Calculus II</u>-style integral.

We'll skip the rest.



$x = \frac{y^2}{4}$

ex:
Type ②

$$\int_C 2xy\,dx + x^2 dy \quad \text{where: } C: \begin{cases} x = 2t \\ y = 3t \end{cases}$$

$0 \le t \le 2$

$$= \int_0^2 \left[2\,x(t)\,y(t)\,\frac{dx}{dt} + x(t)^2 \frac{dy}{dt}\right] dt$$

$\frac{dx}{dt} = 2$

$\frac{dy}{dt} = 3$

$$= \int_0^2 \left[2(2t)(3t)\,2 + (2t)^2\,3\right] dt$$

$$= \int_0^2 \left(24t^2 + 12t^2\right) dt = \int_0^2 36t^2\,dt$$

$$= 12t^3 \Big|_0^2 = 12 \cdot 2^3 - 0 = \boxed{96}$$