16.3 Fundamental Theorem of Line Integrals

Remark: we know that $\vec{F}(x,y,z)$ is a conservative vector field, that is,

$\vec{F}(x,y,z) = \nabla f(x,y,z)$ for some potential function $f(x,y,z)$

then $\text{curl}\,\vec{F} = \vec{0}$

Reason: $\text{curl}\,\vec{F} = \text{curl}\,\nabla f = \vec{0}$ for any real valued function f.

The converse is almost true.

Theorem: If the vector $\vec{F}(x,y,z)$ is defined on all of $\mathbb{R}^3$ then if $\text{curl}\,\vec{F} = 0$ then $\vec{F} = \nabla f$ for some f, that is, F is conservative

Summary: $\vec{F} = \nabla f \Rightarrow \text{curl}\,\vec{F} = \vec{0}$ (always)

$\vec{F} = \nabla f \Leftarrow \text{curl}\,\vec{F} = \vec{0}$ (true if domain of $\vec{F}$ has no holes)

Remark: In $\mathbb{R}^2$ all this is still true, but replace $\vec{F}(x,y,z)$ with $\vec{F}(x,y)$ and replace $\text{curl}\,\vec{F}$ with $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix}$

Note: $\rightarrow$ $= \text{curl}\,\vec{F}(x,y,0) \cdot \hat{k}$

How to find a potential function $f(x,y,z)$ when curl $\vec{F}=0$.

ex ($\vec{F}$ a plane vector field) Given $\vec{F}(x,y) = \langle \underset{P(x,y)}{2xy+1}, \underset{Q(x,y)}{x^2} \rangle$.

Apply the conservative test: Is F conservative?

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 2xy+1 & x^2 \end{vmatrix} = \frac{\partial}{\partial x}[x^2] - \frac{\partial}{\partial y}[2xy+1]$$

$$= 2x - 2x = 0$$ Yes. $\vec{F}$ is conservative.

Let's find $f(x,y)$ so that $\nabla f(x,y) = \vec{F}(x,y)$

"method of matching terms" → [Not the textbook's method] For some $f(x,y)$ (to be determined)

$\nabla f = \vec{F}$ that is $\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \rangle = \langle P, Q \rangle$, so that is

Note: We are not "taking" derivatives of f. We're recognizing some properties off.

$$\begin{cases} \frac{\partial f}{\partial x} = P(x,y) = 2xy+1 \\ \frac{\partial f}{\partial y} = Q(x,y) = x^2 \end{cases}$$

$$\Rightarrow f(x,y) = \int P(x,y)\,dx = \int (2xy+1)\,dx = x^2y + x + g(y)$$

$$f(x,y) = \int Q(x,y)\,dy = \int x^2\,dy = x^2y + h(x)$$

These are the same f

Matching these two expressions, term by term we see:

$$\begin{cases} x^2y = x^2y \\ x = h(x) \\ g(y) = 0 \end{cases}$$

So we can take $\boxed{f(x,y) = x^2y + x}$

Check: $\nabla f(x,y) = \langle 2xy+1, x^2 \rangle$ ✓

ex($\vec{F}$ in $\mathbb{R}^3$) $\vec{F}(x,y,z) = \langle \underbrace{\sin y + yz^2}_{P(x,y,z)},\ \underbrace{x\cos y + xz^2 + z}_{Q(x,y,z)},\ \underbrace{2xyz + y}_{R(x,y,z)} \rangle$

Is $\vec{F}$ conservitive?

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + yz^2 & x\cos y + xz^2 + z & 2xyz + y \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}[2xyz + y] - \frac{\partial}{\partial z}[x\cos y + xz^2 + z] \right) \hat{i}$$

$$+ \left(\frac{\partial}{\partial z}[\sin y + yz^2] - \frac{\partial}{\partial x}[2xyz + y] \right) \hat{j}$$

$$+ \left(\frac{\partial}{\partial x}[x\cos y + xz^2 + z] - \frac{\partial}{\partial y}[\sin y + yz^2] \right) \hat{k}$$

$$= [(2xz + 1) - (2xz + 1)]\, \hat{i}$$

$$+ [(2yz + 1) - (2yz + 1)]\, \hat{j}$$

$$+ [(\cos y + z^2) - (\cos y + z^2)]\, \hat{k} = \vec{0}$$

Yes $\vec{F}$ is conservative.

ex (contd)
Let's find the potential function $f(x,y,z)$.

$$f_x(x,y,z) = P(x,y,z) = \sin y + yz^2$$

$$\Rightarrow f(x,y,z) = \int P\,dx = \int (\sin y + yz^2)\,dx$$
$$= x\sin y + xyz^2 + g(y,z) \quad \boxed{1}$$

$$f_y(x,y,z) = Q(x,y,z) = (x\cos y + xz^2 + z)$$

$$\Rightarrow f(x,y,z) = \int Q\,dy = x\sin y + xyz^2 + yz + h(x,z) \quad \boxed{2}$$

$$f_z(x,y,z) = R(x,y,z) = 2xyz + y$$

$$\Rightarrow f(x,y,z) = \int R\,dz = \int (2xyz + y)\,dz$$
$$= xyz^2 + yz + k(x,y) \quad \boxed{3}$$

Match:
$$x\sin y = x\sin y = k(x,y)$$
$$xyz^2 = xyz^2 = xyz^2$$
$$g(y,z) = yz = yz$$
$$0 = h(x,z) = 0$$

$$f(x,y,z) = x\sin y + xyz^2 + yz$$

[and check that $\nabla f = \vec{F}$]

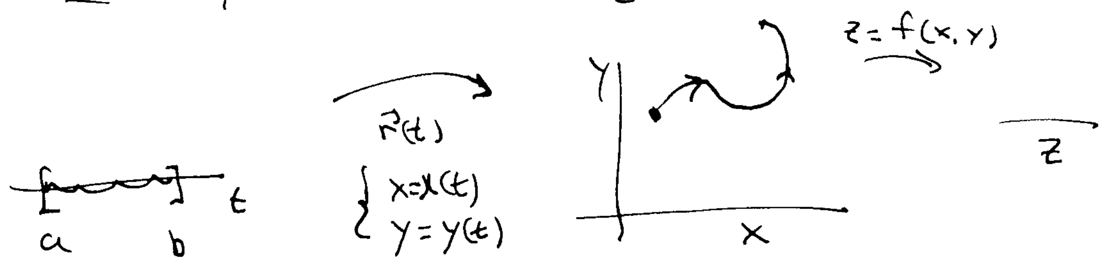
<u>Fundamental Theorem of Line Integrals</u>: Let C be a smooth curve defined by $\vec{r}(t)$ $a \le t \le b$

Equivalently: $\begin{cases} x = x(t) \\ y = y(t) \end{cases}$ $a \le t \le b$

Suppose $\vec{F}(x,y) = \nabla f(x,y)$ for $f(x,y)$.

Then
$$\int_C \vec{F} \cdot dr = \int_C P\,dx + Q\,dy$$
$$= f(\vec{r}(b)) - f(\vec{r}(a))$$
$$= f(x(b), y(b)) - f(x(a), y(a))$$

<u>Idea</u> of proof: Chain Rule



$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = P(x(t), y(t))\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt}$$

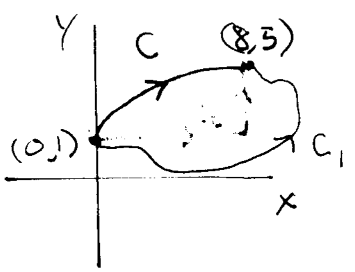
$$\int_C P\,dx + Q\,dy = \int_a^b \left[P(x(t), y(t))\,\frac{dx}{dt} + Q(x(t), y(t))\frac{dy}{dt}\right] dt$$
$$= \int_a^b \frac{d}{dt}\left[f(x(t), y(t))\right] dt$$
$$= f(x(b), y(b)) - f(x(a), y(a)).$$

ex: For $\vec{F}(x,y) = \langle \underset{P(x,y)}{2xy+1}, \underset{Q(x,y)}{x^2} \rangle$

find $\int_C \vec{F} \cdot d\vec{r} = \int_C P\,dx + Q\,dy$

$$= \int_C (2xy+1)\,dx + x^2\,dy$$

where $C: \begin{cases} x = t^3 \\ y = t^2+1 \end{cases}$ $0 \le t \le 2$



when $t=0$, $(x,y) = (0,1)$
$t=2$ $(x,y) = (8,5)$

and $t = x^{1/3}$ so $y = x^{2/3}+1$

From earlier work $\vec{F} = \nabla f$ where $f(x,y) = x^2y + x$.

$$\int_C (2xy+1)\,dx + x^2\,dy = f(8,5) - f(0,1)$$

$$= (8^2 \cdot 5 + 8) - (0^2 \cdot 1 + 0)$$

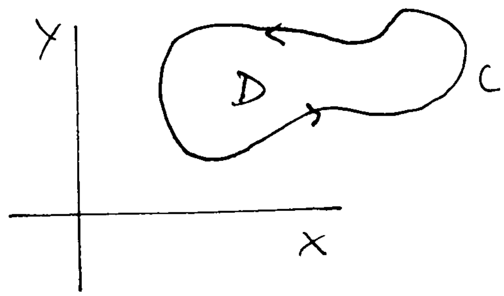
$$= 320 + 8 = 328$$

Important remark: Our answer depended only on the beginning point and the endpoint of C
"The integral is independent of path."

Fact: $\int_C \vec{F} \cdot d\vec{r}$ is independent of path $\Leftrightarrow$ $\oint_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve C.

[Read about this]

16.4 Green's Theorem



"positively oriented"

[think : right hand rule]

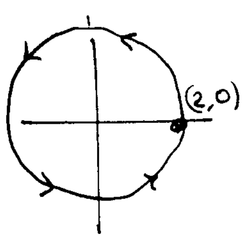
closed curve

that is, C is the boundary of region D.

$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

ex. Find $\displaystyle\int_C \overbrace{(x^2-y)}^{P}\,dx + \overbrace{(x+y^2)}^{Q}\,dy = (\ast)$

where $C: \begin{cases} x = 2\cos t \\ y = 2\sin t \end{cases}$ $\quad 0 \le t \le 2\pi$



by applying Green's theorem.

$$\begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{vmatrix} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{\partial}{\partial x}[x+y^2] - \frac{\partial}{\partial y}[x^2-y]$$

$$= 1 - (-1) = 2 \ [\neq 0]$$

$$\text{So } (\ast) = \iint_D 2\,dA = 2\iint_D dA = 2\,[\text{area of circle of radius } 2]$$

$$= 2\,[\pi 2^2] = \boxed{8\pi}$$

because $\displaystyle\iint_D 2\,dA = \int_0^{2\pi}\int_0^2 2r\,dr\,d\theta = \int_0^2 2r\,dr \cdot \int_0^{2\pi} d\theta = \left[r^2\right]_0^2 \left[\theta\right]_0^{2\pi} = 4\cdot 2\pi = 8\pi$