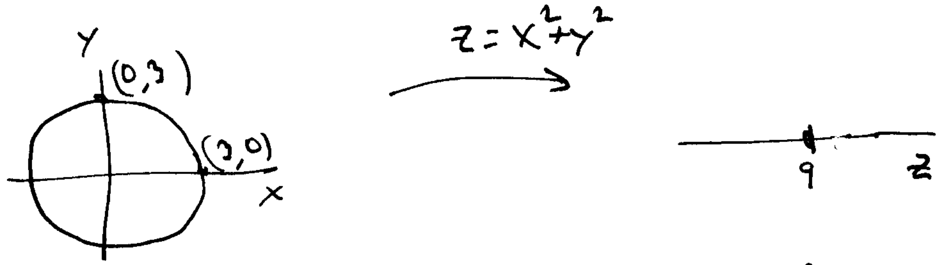
Math 281 Wed 18 May 2016

16.6 Surface area and parametric surfaces

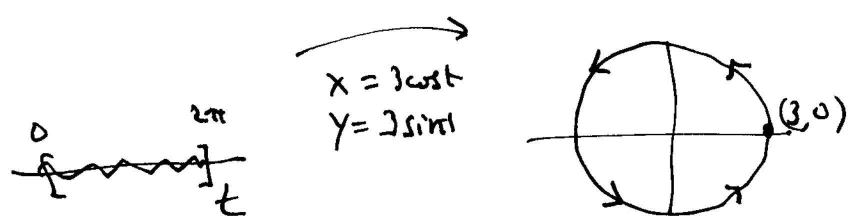
ex: How can we represent a circle of radius 3 and center (0,0)?

a) As $\{x,y \mid x^2+y^2=9\}$, an implicitly-defined curve.



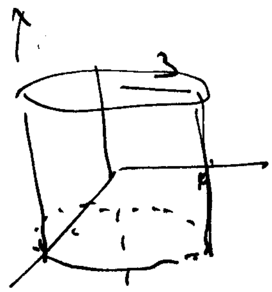
Note: that this a level curve of $f(x,y)=x^2+y^2$.

b) As the parametric curve $\begin{cases} x = 3\cos t \\ y = 3\sin t \end{cases}$ $0 \le t \le 2\pi$

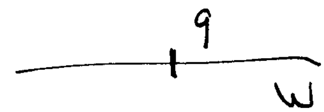


ex: How can we represent a cylinder of radius 3, axis = z-axis.

a) As $\{(x, y, z) \mid x^2+y^2 = 9\}$

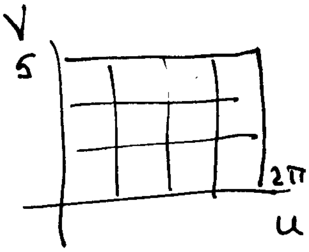


$\xrightarrow{w = x^2+y^2}$

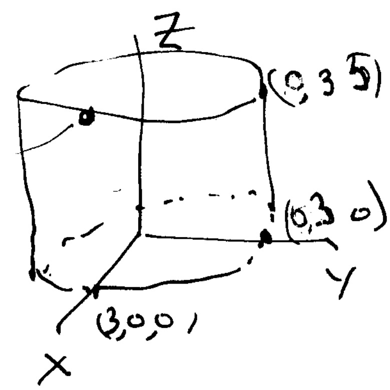


Note: This a level surface of $f(x,y,z) = x^2+y^2$.

b) As a parametric surface



$\longrightarrow$



$$\begin{cases} x(u,v) = 3\cos u \\ y(u,v) = 3\sin u \\ z(u,v) = v \end{cases}$$

$0 \le u \le 2\pi$

$0 \le v \le 5$

How do we know what the surface is given parametric equations in u and v?

Methods ①: Eliminate parameters

or ②: Trace the parametric curves produced by holding one parameter fixed.

ex ①: $x = 3\cos u \Rightarrow x^2 = 9\cos^2 u$

$y = 3\sin u \qquad y^2 = 9\sin^2 u$

$x^2 + y^2 = 9$

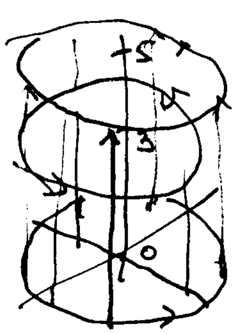
ex 2: in $\left\{ \begin{aligned} x &= 3\cos u \\ y &= 3\sin u \\ z &= v \end{aligned} \right.$

Set $v = 3$, Let $0 \le u \le 2\pi$

Get $x(u,3) = 3\cos u$

$y(u,3) = 3\sin u$

$z(u,3) = 3$



$\left(\frac{3\sqrt{2}}{2}, \frac{3\sqrt{2}}{2}, 0\right)$

Now, hold $u = \pi/4$, let $0 \le v \le 5$

$x(\pi/4, v) = 3\cos \pi/4 = 3 \cdot \frac{\sqrt{2}}{2}$

$y(\pi/4, v) = 3\sin \pi/4 = 3 \cdot \sqrt{2}/2$

$z(\pi/4, v) = v$

Remark ① The parametric curve $\begin{cases} x(t) = 3\cos t \\ y(t) = 3\sin t \end{cases}$ $0 \le t \le \pi$

is equivalent to the vector function

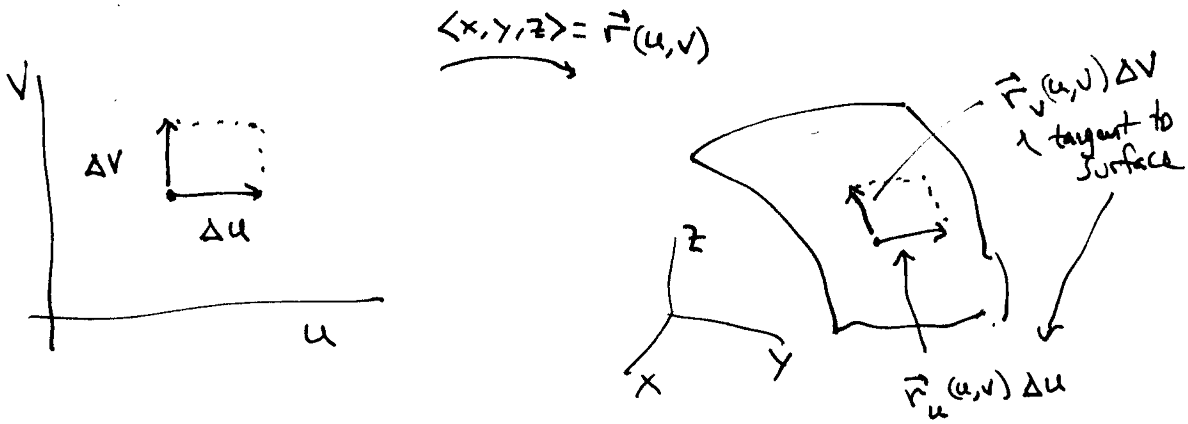
$$\vec{r}(t) = \langle 3\cos t, 3\sin t \rangle$$

② the parametric surface $\begin{cases} x(u,v) = 3\cos u \\ y(u,v) = 3\sin u \\ z(u,v) = v \end{cases}$

is equivalent to the vector function of two variables

$$\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v \rangle$$

Element of area = dS, for parametric surfaces.



$(\vec{r}_u \Delta u \times \vec{r}_v \Delta v)$ will be vector orthogonal to both vectors, hence normal to the surface.

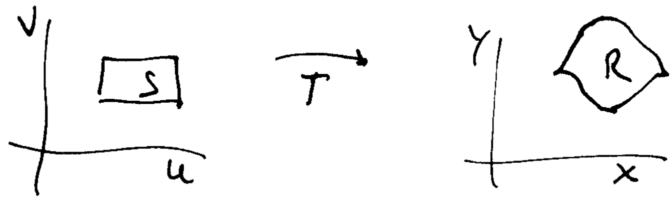
Also $dS = |\vec{r}_u \Delta u \times \vec{r}_v \Delta v| = |\vec{r}_u \times \vec{r}_v| \Delta u \Delta v$ = area of the parallelogram spanned the two tangent vectors.

Defn.

$$\boxed{dS = |\vec{r}_u(u,v) \times \vec{r}_v(u,v)|\, du\,dv}$$

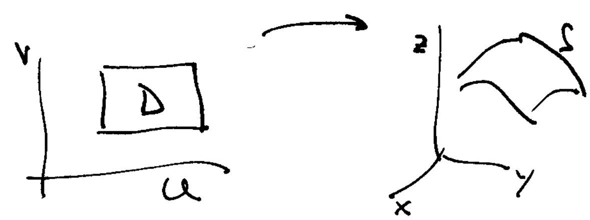
defines the element of surface area for the parametric surface defined by $\vec{r}(u,v) = \langle x, y, z\rangle$.

Remark: This plays ~~pla~~ the same role as the Jacobian in a change of variables in the plane.



How to calculate area of a parametric surface:

$$\text{area of surface} = \iint_S dS = \iint_D |r_u \times r_v|\, du\,dv$$



example (cont'd) If $\vec{r}(u,v) = \langle 3\cos u, 3\sin u, v\rangle$ then

$$\vec{r}_u(u,v) \times \vec{r}_v(u,v) = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ -3\sin u & 3\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 3\cos u, 3\sin u, 0\rangle$$

$$|\vec{r}_u \times \vec{r}_v| = \sqrt{(3\cos u)^2 + (3\sin u)^2 + 0^2} = \sqrt{9(\cos^2 u + \sin^2 u)} = 3$$

$$\text{surface area} = \iint_S dS = \iint_D |\vec{r}_u \times \vec{r}_v|\, du\,dv = \int_0^5 \int_0^{2\pi} 3\, du\,dv = 3[u]_0^{2\pi}[v]_0^5 = 3(2\pi)(5) = \boxed{30\pi}$$